

A limit theorem to a time-fractional diffusion

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October 5, 2011

Abstract

We prove a limit theorem for an integral functional of a Markov process. The Markovian dynamics is characterized by a linear Boltzmann equation modeling a one-dimensional test particle of mass $\lambda^{-1} \gg 1$ in an external periodic potential and undergoing collisions with a background gas of particles with mass one. The object of our limit theorem is the time integral of the force exerted on the test particle by the potential, and we consider this quantity in the limit that λ tends to zero for time intervals on the scale λ^{-1} . Under appropriate rescaling, the total drift in momentum due to the potential converges to a Brownian motion, time-changed by the local time at zero of an Ornstein-Uhlenbeck process.

1 Introduction

1.1 Model and results

Consider the family $\lambda \in \mathbb{R}^+$ of Markov processes $(X_t^{(\lambda)}, P_t^{(\lambda)}) \in \mathbb{R}^2$ whose densities $\Psi_{t,\lambda}(x, p)$ obey the forward Kolmogorov equation

$$\begin{aligned} \frac{d}{dt}\Psi_{t,\lambda}(x, p) &= -p\frac{\partial}{\partial x}\Psi_{t,\lambda}(x, p) + \frac{dV}{dx}(x)\frac{\partial}{\partial p}\Psi_{t,\lambda}(x, p) \\ &\quad + \int_{\mathbb{R}} dp' (\mathcal{J}_{\lambda}(p', p)\Psi_{t,\lambda}(x, p') - \mathcal{J}_{\lambda}(p, p')\Psi_{t,\lambda}(x, p)), \end{aligned} \quad (1.1)$$

where $V(x) = V(x+1) \geq 0$ is continuously differentiable, and the jump kernel $\mathcal{J}_{\lambda}(p, p')$ has the form

$$\mathcal{J}_{\lambda}(p', p) = \frac{(1+\lambda)}{64}|p' - p|e^{-\frac{1}{2}\left(\frac{1-\lambda}{2}p' - \frac{1+\lambda}{2}p\right)^2}. \quad (1.2)$$

The values $\mathcal{J}_{\lambda}(p', p)$ correspond to the rate of jumps from (x, p') to (x, p) . The Kolmogorov equation above is an idealized description of the phase space density for a test particle in dimension one which feels a spatially periodic force $\frac{dV}{dx}(x)$ and receives elastic collisions with

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particles from a gas. The jump rates \mathcal{J}_λ correspond to the one-dimensional case of equation (8.118) from [24], in which the mass of a single reservoir particle is set to one, the temperature of the gas is set to one, the spatial density of the gas is set to $\frac{1}{32}(2\pi)^{\frac{1}{2}}$, and the mass of the test particle is λ^{-1} .

We will subsequently suppress the λ -dependence of the dynamics by removing the superscript for the process: (X_t, P_t) . The cumulative drift D_t in the particle's momentum up to time t due to the periodic force field has the form

$$D_t = \int_0^t dr \frac{dV}{dx}(X_r).$$

The momentum at time t can be written in the form $P_t = P_0 + D_t + J_t$, where J_t is the sum of all the momentum jumps due to collisions with the gas. To state our main result contained in Theorem 1.1 below, let us define the limiting processes. Define $\mathbf{p} \in \mathbb{R}$ to be the process satisfying the Langevine equation

$$d\mathbf{p}_t = -\frac{1}{2}\mathbf{p}_t dt + d\mathbf{B}'_t, \quad (1.3)$$

where \mathbf{B}' is a standard Brownian motion. The solution \mathbf{p} is referred to as the Ornstein-Uhlenbeck process [26]. Moreover, let the process \mathbf{l} denote the local time at zero for the process \mathbf{p} . Recall that the local time at a point $a \in \mathbb{R}$ over the interval $[0, t]$ is formally given by the expression: $\int_0^t dr \delta_a(\mathbf{p}_r)$.

In [6], it was shown that $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$ converges in law to \mathbf{p} over any finite time interval $[0, T]$, and that the expectation of $\sup_{0 \leq t \leq T} |\lambda^{\frac{1}{4}}D_{\frac{t}{\lambda}}|$ is uniformly bounded for all $\lambda < 1$. Theorem 1.1 extends this result to a limit law for $\lambda^{\frac{1}{4}}D_{\frac{t}{\lambda}}$ which is joint with that of $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$. The rescaled momentum drift $\lambda^{\frac{1}{4}}D_{\frac{t}{\lambda}}$ converges to a diffusion process, time-changed by the local time of the Ornstein-Uhlenbeck process \mathbf{p} that $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$ limits to.

Theorem 1.1. *Assume that $V(x)$ is continuously differentiable and that the initial distribution μ has finite moments in momentum: $\int_{\mathbb{R}^2} d\mu(x, p) |p|^m < \infty$ for $m \geq 1$. In the limit $\lambda \rightarrow 0$, there is convergence in law of the process pair*

$$\left(\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}, \lambda^{\frac{1}{4}}D_{\frac{t}{\lambda}} \right) \xrightarrow{\mathcal{L}} (\mathbf{p}_t, \sqrt{\kappa}\mathbf{B}_{\mathbf{l}_t}) \quad t \in [0, T],$$

for a constant $\kappa > 0$, and where \mathbf{l} is the local time at zero of \mathbf{p} , and \mathbf{B} is a copy of Brownian motion which is independent of \mathbf{p} . The convergence is with respect to the Skorokhod metric.

The diffusion constant κ is formally given by a Green-Kubo form which is remarked on in Section 1.2.

Theorem 1.1 implies that the contribution J_t to the momentum due to collisions has higher order than the forcing part D_t . In particular, $\lambda^{\frac{1}{2}}J_{\frac{t}{\lambda}}$ converges to the Ornstein-Uhlenbeck process as $\lambda \rightarrow 0$. In the conjecture below, we give a more refined statement for the limiting law of the full momentum $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$ for small λ , which takes into account the perturbative contribution of the forcing term $\lambda^{\frac{1}{2}}D_{\frac{t}{\lambda}}$. In this approximation, the contribution of the periodic force is given by a diffusive pulse that the momentum feels when it returns to the region around the value zero. The \mathbf{p} in the statement of the conjecture should be thought of as the limit in law of the collision contribution $\lambda^{\frac{1}{2}}J_{\frac{t}{\lambda}}$.

Conjecture 1.2. *Make the assumptions of Theorem 1.1, and let $F : C([0, T]) \rightarrow \mathbb{C}$ be bounded and smooth with respect to the supremum norm. Define the process $\mathbf{p}_{t,\lambda}$ as*

$$\mathbf{p}_{t,\lambda} = \mathbf{p}_t + \sqrt{\kappa} \lambda^{\frac{1}{4}} \left(\mathbf{B}_{\mathbf{l}_r} - \frac{1}{2} \int_0^t dr e^{-\frac{1}{2}(t-r)} \mathbf{B}_{\mathbf{l}_r} \right), \quad (1.4)$$

where \mathbf{p} , \mathbf{B} , \mathbf{l} , and $\kappa > 0$ are defined as in Theorem 1.1. Then the law of the process $\lambda^{\frac{1}{2}} P_{\frac{t}{\lambda}}$ is close to the law of $\mathbf{p}_{\cdot,\lambda}$ for $\lambda \ll 1$ in the sense that

$$\mathbb{E}[F(\lambda^{\frac{1}{2}} P_{\frac{t}{\lambda}})] = \mathbb{E}[F(\mathbf{p}_{\cdot,\lambda})] + O(\lambda^{\frac{1}{2}}).$$

Note that if $\mathbf{p}_{t,\lambda}$ is replaced by $\mathbf{p}_{t,0} = \mathbf{p}_t$ in the expectation above, then the error can at best be $O(\lambda^{\frac{1}{4}})$.

1.2 Discussion

Theorem 1.1 characterizes the limiting law for the integral functional of the Markov process $S_t = (X_t, P_t)$ given by

$$D_t = \int_0^t dr g(S_r), \quad g(x, p) = \frac{dV}{dx}(x), \quad (1.5)$$

for time scales $t \propto \lambda^{-1}$ and normalization factor $\lambda^{\frac{1}{4}}$. The underlying law of the process S_t depends on the parameter λ through the jump rate kernel \mathcal{J}_λ . Since the potential $V(x)$ has period one, it is convenient to view S_t as having state space $\Sigma = \mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = [0, 1]$ is the unit torus, rather than \mathbb{R}^2 . The process $S_t \in \Sigma$ is ergodic to an equilibrium state given by the Maxwell-Boltzmann distribution

$$\Psi_{\infty,\lambda}(x, p) = \frac{e^{-\lambda H(x,p)}}{N(\lambda)}, \quad (1.6)$$

where $H(x, p) = \frac{1}{2}p^2 + V(x)$ and for a normalization constant $N(\lambda)$. Although the ergodicity is exponential in nature, the rate of ergodicity decays as λ goes to zero, and thus, a limit theorem for a normalized version of $D_{\frac{t}{\lambda}}$ does not fall under the limit theory for integral functionals of an ergodic Markov process [16]. This is also clear from the appropriate scaling factor of $D_{\frac{t}{\lambda}}$ being $\lambda^{\frac{1}{4}}$ rather than $\lambda^{\frac{1}{2}}$. Heuristics for this scaling were given in [6, Sec. 1.2.2], and the smaller exponent for the scaling is driven by the fact that $\frac{dV}{dx}(X_r)$ is typically oscillating with high frequency ($\propto \lambda^{-\frac{1}{2}}$) around zero for most of the time interval $[0, \frac{T}{\lambda}]$. These oscillations in $\frac{dV}{dx}(X_r)$ occur as the particle revolves around the torus with speed $|P_r|$, which typically is found on the order $\lambda^{-\frac{1}{2}}$. The fluctuations in D_t have a chance to accumulate primarily when $|\lambda^{\frac{1}{2}} P_{\frac{t}{\lambda}}|$ dips down to “small” values, and this suggests that a rescaled version of $D_{\frac{t}{\lambda}}$ should converge in law to the local time at zero for the limiting law of $\lambda^{\frac{1}{2}} P_{\frac{t}{\lambda}}$.

As $\lambda \rightarrow 0$, the jump rates approach the form $\mathcal{J}_0(p, p') = j(p - p')$ for

$$j(p) = \frac{1}{64}|p|e^{-\frac{1}{8}p^2}, \quad (1.7)$$

which describe a random walk in momentum. Thus the process S_t behaves more like a null-recurrent Markov process for small λ . This idea breaks down at time-scales $\propto \lambda^{-1}$ where

a first-order contribution to $\mathcal{J}_\lambda(p, p')$ around $\lambda = 0$ generates the frictional drag to smaller momenta seen in the linear drift term of the Langevin equation (1.3) which defines \mathbf{p}_t . The diffusion constant κ in Theorem 1.1 is formally given by the Green-Kubo expression

$$\kappa = 2 \int_{[0, 1] \times \mathbb{R}} dx dp \frac{dV}{dx}(x) \mathfrak{R}^{(0)}\left(\frac{dV}{dx}\right)(x, p), \quad (1.8)$$

where $\mathfrak{R}^{(0)} = \int_0^\infty dr e^{r\mathcal{L}_0}$ is the reduced resolvent of the backwards generator \mathcal{L}_0

$$(\mathcal{L}_0 F)(x, p) = p \frac{\partial}{\partial x} F(x, p) - \frac{dV}{dx}(x) \frac{\partial}{\partial p} F(x, p) + \int_{\mathbb{R}} dp' j(p') (F(x, p + p') - F(x, p)),$$

where $F \in L^\infty$ is differentiable.

The null-recurrent behavior for the process $S_t = (X_t, P_t)$ emerging as $\lambda \rightarrow 0$ at short time scales, and the relaxation behavior which takes place on time scales $\propto \lambda^{-1}$ are both apparent in the limiting law $\sqrt{\kappa} \mathbf{B}_{\mathfrak{l}_t}$; the diffusion constant κ is defined in terms of the jump rates (1.7) which correspond to an unbiased random walk, and on the other hand, the local time process \mathfrak{l}_t is defined in terms of the Ornstein-Uhlenbeck process which has exponential relaxation (in the correct norm) to the Maxwell-Boltzmann distribution $(\frac{1}{2\pi})^{\frac{1}{2}} e^{-\frac{1}{2}q^2}$.

1.2.1 The limiting processes

As before, we let \mathfrak{l} be the local time of the Ornstein-Uhlenbeck process \mathbf{p} and \mathbf{B} be a standard Brownian motion independent of \mathbf{p} . Recall that the local time process $\mathfrak{l}^{(a)}$ for a point $a \in \mathbb{R}$ is the a.s. continuous increasing process formally given by

$$\mathfrak{l}_t^{(a)} = \int_0^t dr \delta_a(\mathbf{p}_r).$$

For each realization of the process \mathbf{p} over the interval $[0, t]$, $\mathfrak{l}_t^{(a)}$ is the density of time that the path for \mathbf{p} spends at a , and thus $\int_{\mathbb{R}} da \mathfrak{l}_t^{(a)} = t$. For the case $a = 0$, we neglect the superscript for $\mathfrak{l}^{(a)}$. The values of \mathfrak{l} stay fixed over the time intervals in which \mathbf{p} moves away from the origin, and thus, in a sense, \mathfrak{l} makes its increases over the set of times with Hausdorff dimension $\frac{1}{2}$ where $\mathbf{p}_t = 0$. The fractional diffusion process $\sqrt{\kappa} \mathbf{B}_{\mathfrak{l}}$, appearing as the $\lambda \rightarrow 0$ limit in law of $\lambda^{\frac{1}{4}} D_{\frac{1}{\lambda}}$ in Theorem 1.1, has its fluctuations constrained to those times in which \mathfrak{l} increases. Clearly, $\sqrt{\kappa} \mathbf{B}_{\mathfrak{l}}$ is not Markovian, since the amount time that the process $\sqrt{\kappa} \mathbf{B}_{\mathfrak{l}}$ has held its current value (i.e. the excursion time of \mathbf{p} from zero) is correlated with the amount time that it is likely to remain fixed at that value. The densities $\rho_t : \mathbb{R} \rightarrow \mathbb{R}^+$ of $\sqrt{\kappa} \mathbf{B}_{\mathfrak{l}_t}$ satisfy the Volterra-type integro-differential equation

$$\rho_t(q) = \rho_0(q) + \frac{\kappa}{2(2\pi)^{\frac{1}{2}}} \int_0^t dr \frac{(\Delta_q \rho_r)(q)}{\left(1 - e^{-\frac{1}{2}(t-r)}\right)^{\frac{1}{2}}}, \quad \rho_0(q) = \delta_0(q). \quad (1.9)$$

The non-Markovian nature of the processes $\sqrt{\kappa} \mathbf{B}_{\mathfrak{l}}$ is visible in the convolution form in (1.9). The master equation above is similar to the master equation for a Brownian motion with diffusion constant κ time-changed by an independent Mittag-Leffler process $\mathfrak{m}^{(\alpha)}$ of index $0 < \alpha < 1$. Note that our limiting processes does not satisfy any scale invariance, since \mathbf{p} does not and thus \mathfrak{l} does not. Some further discussion of local time and related material is included in Appendix A.

1.2.2 Related literature

The limit theory for integral (or summation) functionals of Markov processes (respectively, chains) usually splits into several standard categories depending on whether the limiting procedure is of first- or second-order and whether the Markov process is positive-recurrent or null-recurrent. Second-order limit theorems for integral functionals of ergodic Markov processes are well-understood (for instance [15], and see the book [16] for a broader discussion of the literature). In the null-recurrent case, second-order limit theory for integral functionals is discussed in [25], in [22, 9] when the Markov process is a diffusion, and in [4] for a Markov chain rather than a process. The second-order theory is closely related to the limit theory for martingales by a standard construction (1.10) which seems to have been introduced in [11] (in the analogous case of a chain). Limit results for martingales with quadratic variations which are additive functionals of null-recurrent Markov processes can be found in [25, 12]. That literature builds on and applies the limit theory for additive functionals of Markov processes (see, for instance, [3, 8] and for more recent results [17, 18]) which began with a paper by Darling and Kac [10]. The monograph [12] is a particularly useful reference on the subject, which, in addition to presenting new results, serves some purpose as a review.

The usual recipe for finding a martingale close to an integral functional $\int_0^t dr g(S_r)$ of a Markov process is given by the following: if S_t is a Harris recurrent Markov process and g is a function defined on its state space such that the reduced resolvent \mathfrak{R} of the backward evolution operating on g is “well-behaved” (e.g. lives in a suitable L^p space), then

$$\tilde{M}_t = (\mathfrak{R}g)(S_t) - (\mathfrak{R}g)(S_0) + \int_0^t dr g(S_r) \quad (1.10)$$

is a martingale. The difference between $\int_0^t dr g(S_r)$ and \tilde{M}_t is a pair of terms which are comparatively small in many situations. For our model, it is not clear for us how to obtain the necessary bounds on the reduced resolvent $(\mathfrak{R}^{(\lambda)} \frac{dV}{dx})(s)$ in the limit $\lambda \rightarrow 0$ to exploit (1.10), and we use a variant of this martingale (see Lemma 2.1). To build a martingale approximating D_t , we expand the state space from Σ to $\tilde{\Sigma} = \Sigma \times \{0, 1\}$ using a Nummelin splitting-type construction. The benefit of viewing the process in the extended state space is that the trajectories for the process S_t can be decomposed into a series of nearly i.i.d. parts corresponding to time intervals $[R_n, R_{n+1})$ where R_n are associated with the return times to an “atom” identified with the subset $\Sigma \times 1 \subset \tilde{\Sigma}$. This allows the integral functional D_t to be written as a pair of boundary terms plus a random sum of nearly i.i.d. random variables.

We briefly discuss the history of these splitting techniques. For Markov chains, a technique for extending the dynamics from a state space Σ to $\Sigma \times \{0, 1\}$ in order to embed an atom was developed independently in [21] and [1], and this is referred to as *Nummelin splitting* or merely *splitting*. When it comes to the splitting of Markov processes, there are different schemes offered in [12] and [17]. In [12], there is a sequence of split processes constructed which contain marginal processes that are arbitrarily close to the original process. The construction in [17] involves a larger state space $\Sigma \times [0, 1] \times \Sigma$, although an exact copy of the original process is embedded as a marginal. The splitting construction that we employed in [6] and use in the current article is a truncated version of that in [17], although the split process that we consider is not Markovian due to the truncation. The idea of applying splitting techniques to obtain limit theorems for integral functionals of null-recurrent Markov processes was introduced in [25] and has been developed further in other limit theory in [3, 4, 12].

There are some basic differences that should be emphasized between our model and models for the results mentioned above. The law for our underlying Markovian process S_t is itself λ -dependent. This is not the case for the limit theorems discussed above in which there is a single fixed Markovian dynamics, and a parameter λ only appears in the length of the time intervals considered and in the scaling factors for the variables of interest. This is why it is possible for us to get a limit law $\sqrt{\kappa}\mathbf{B}_t$ which has no scale invariance. The limit theorems for integral functionals $\int_0^t dr g(S_r)$ of null-recurrent Markov processes considered in [25, 22, 9] assume that the “velocity function” g exists in L^1 with respect to the invariant measure of the process. This effectively means that the null-recurrent process S_t spends most of the time in regions of phase space where $g(S_t)$ is “small”. In our case, the function $g(x, p) = \frac{dV}{dx}(x)$ has no decay as $|p| \rightarrow \infty$, and we rely on the rapid oscillations of $\frac{dV}{dx}(X_r)$ which occur when $|P_r| \gg 1$.

Our techniques could be used to prove analogous results for a related model in [7]. In that case, the limiting law for a rescaling of the pair (P_t, D_t) (momentum and integral of the force) would have the form $(\sqrt{\sigma}\mathbf{B}', \sqrt{\kappa}\mathbf{B}_l)$ for some $\sigma, \kappa > 0$, where \mathbf{B}' , \mathbf{B} are independent copies of standard Brownian motion, and l is the local time at zero for \mathbf{B}' .

1.2.3 Comments on Conjecture 1.2

Conjecture 1.2 characterizes the perturbative influence for $\lambda \ll 1$ on the momentum of the particle when the periodic force is turned on. The process $\mathbf{p}_{t,\lambda}$ formally satisfies the Langevin equation

$$d\mathbf{p}_{t,\lambda} = -\frac{1}{2}\mathbf{p}_{t,\lambda} dt + d\mathbf{B}'_t + \lambda^{\frac{1}{4}}\sqrt{\kappa}\delta_0(\mathbf{p}_t)d\mathbf{B}''_t, \quad (1.11)$$

where $\mathbf{p}_{0,\lambda} = 0$, \mathbf{B}' and \mathbf{p} are defined as in (1.3), and \mathbf{B}'' is a copy of standard Brownian motion independent of \mathbf{B}' . This makes the identification $\int_0^t d\mathbf{B}''_r \delta_0(\mathbf{p}_r) = \mathbf{B}_l$. Through equation (1.11), $\mathbf{p}_{t,\lambda}$ has the appearance of what would be a first-order approximation for $\lambda \ll 1$ of a processes $\mathbf{p}'_{t,\lambda}$ satisfying the stochastic differential equation

$$d\mathbf{p}'_{t,\lambda} = -\frac{1}{2}\mathbf{p}'_{t,\lambda} dt + d\mathbf{B}'_t + \lambda^{\frac{1}{4}}\sqrt{\kappa}\delta_0(\mathbf{p}'_{t,\lambda})d\mathbf{B}''_t.$$

However, this equation can not be made sensible.

1.3 Organization of the article

Section 2 outlines the construction of a version of the process $S_t = (X_t, P_t)$ in an enlarged state space. Section 3 contains the proof of Theorem 3.1, which effectively makes the connection between the normalized momentum process $\lambda^{\frac{1}{2}}P_{\frac{1}{\lambda}}$ and the local time l appearing in the limiting law for $\lambda^{\frac{1}{4}}D_{\frac{1}{\lambda}}$. Section 4 contains a formulation of the “martingale problem” which determines the uniqueness of the limiting law $(\mathbf{p}, \sqrt{\kappa}\mathbf{B}_l)$ in the proof of Theorem 1.1. The proof of Theorem 1.1 is in Section 5, and Appendix A contains some discussion of the limit process \mathbf{B}_l . We will make the assumptions of Theorem 1.1 throughout the text.

2 Nummelin splitting

We will now summarize the particular splitting structure defined in [6, Sec. 4.1] which extends the state space of the process. This construction is contained in the first two components of

the split process introduced in [17]. The result is a process which behaves nearly as though the state space contains a recurrent atom. This has the advantage that the life cycles between returns to the “atom” are nearly uncorrelated. To do this, we first introduce a resolvent chain embedded in the original process. We then split the chain using the standard technique [1, 21], and we extend the resolvent chain to a non-Markovian process which contains an embedded version of the original process.

Let e_m , $m \in \mathbb{N}$ be mean-1 exponential random variables which are independent of each other and of the process $S_t = (X_t, P_t) \in \Sigma$. Define $\tau_n := \sum_{m=1}^n e_m$, and by convention, we set $\tau_0 = 0$. The τ_n will be referred to as the *partition times*. Define \mathbf{N}_t to be the number of non-zero τ_n less than t , and the Markov chain $\sigma_n = (X_{\tau_n}, P_{\tau_n}) \in \Sigma$, which is referred to as the *resolvent chain*. The transition kernel \mathcal{T}_λ for the chain (i.e. acting on functions from the left and on measures from the right) has the form

$$\mathcal{T}_\lambda = \int_0^\infty dr e^{-r+r\mathcal{L}_\lambda} = (1 - \mathcal{L}_\lambda)^{-1},$$

where \mathcal{L}_λ is the backward Markov generator for the process. The resolvent chain has the same invariant probability density (1.6) as the original process. By Nummelin splitting, which we outline presently, the state space Σ is extended to $\tilde{\Sigma} = \Sigma \times \{0, 1\}$ in order to construct a chain $(\tilde{\sigma}_n) \in \tilde{\Sigma}$ with a *recurrent atom* and such that the statistics for (σ_n) are embedded in the first component of $(\tilde{\sigma}_n)$. For a Markov chain, an *atom* is a subset of the state space such that the transition measure is independent of the element within the subset. The atom is *recurrent* if the event of returning to the atom in the future has probability one.

A probability measure ν on Σ paired with a non-zero function $h : \Sigma \rightarrow [0, 1]$ are said to satisfy the *minorization condition* with respect to \mathcal{T}_λ if

$$\mathcal{T}_\lambda(s_1, ds_2) \geq h(s_1)\nu(ds_2). \quad (2.1)$$

By Part (1) of [6, Prop. 4.3], there exists a $\mathbf{u} > 0$ such that

$$h(s) = \mathbf{u} \frac{\chi(H(s) \leq l)}{U} \quad \text{and} \quad \nu(ds) = ds \frac{\chi(H(s) \leq l)}{U}, \quad (2.2)$$

satisfy the minorization condition, where $l = 1 + 2 \sup_x V(x)$, $U > 0$ is the normalization constant of ν , and $H(x, p) = \frac{1}{2}p^2 + V(x)$. The specific choice of h and ν satisfying (2.1) is not important in this section, although we will take them to be defined as in (2.2) for future sections.

We define the following forward transition operator $\tilde{\mathcal{T}}_\lambda$, which sends the state $(s_1, z_1) \in \tilde{\Sigma}$ to the infinitesimal region (ds_2, z_2) with measure:

$$\tilde{\mathcal{T}}_\lambda(s_1, z_1; ds_2, z_2) = \begin{cases} \frac{1-h(s_2)}{1-h(s_1)}(\mathcal{T}_\lambda - h \otimes \nu)(s_1, ds_2) & z_1 = z_2 = 0, \\ \frac{h(s_2)}{1-h(s_1)}(\mathcal{T}_\lambda - h \otimes \nu)(s_1, ds_2) & z_1 = 1 - z_2 = 0, \\ (1 - h(s_2))\nu(ds_2) & z_1 = 1 - z_2 = 1, \\ h(s_2)\nu(ds_2) & z_1 = z_2 = 1. \end{cases}$$

Given a measure μ on Σ , we refer to its *splitting* $\tilde{\mu}$ as the measure on $\tilde{\Sigma}$ given by

$$\tilde{\mu}(ds, z) = \chi(z = 0)(1 - h(s))\mu(ds) + \chi(z = 1)h(s)\mu(ds). \quad (2.3)$$

In particular, the split chain is taken to have initial distribution given by the splitting of the initial distribution for the original (pre-split) chain. The invariant measure for the chain $(\tilde{\sigma}_n)$

is the splitting $\tilde{\Psi}_{\infty,\lambda}$ of the Maxwell-Boltzmann distribution defined in (1.6). The split chain is positive-recurrent for any $\lambda > 0$, since the original process is positive-recurrent (and, in fact, exponentially ergodic to $\Psi_{\infty,\lambda}$ by [6, Thm. A.1]). The jump rates from $(s_1, 1)$ are independent of $s_1 \in \Sigma$, and thus the set $\Sigma \times 1 \subset \tilde{\Sigma}$ is an atom. The atom is recurrent, since the original chain is positive-recurrent with stationary state $\Psi_{\infty,\lambda}$ and $\tilde{\Psi}_{\infty,\lambda}(\Sigma \times 1) = \Psi_{\infty,\lambda}(h) > 0$. Notice that according to the above transition rates, the probability that $z_2 = 1$ is $h(s_2)$ when given s_1, s_2 , and z_1 .

Using the law for the split chain $\tilde{\sigma}_n \in \tilde{\Sigma}$ determined by the transition rates $\tilde{\mathcal{T}}_\lambda$ above, we may construct a split process $(\tilde{S}_t) \in \tilde{\Sigma}$ and a sequence of times $\tilde{\tau}_n$ with the recipe below. The $\tilde{\tau}_n$ should be thought of as the partition times τ_n embedded in the split statistics, although we temporarily denote them with the tilde to emphasize their axiomatic role in the construction of the split process. Let $\tilde{\tau}_n$ and $\tilde{S}_t = (S_t, Z_t)$ be such that

1. $0 = \tilde{\tau}_0$, $\tilde{\tau}_n \leq \tilde{\tau}_{n+1}$, and $\tilde{\tau}_n \rightarrow \infty$ almost surely.
2. The chain $(\tilde{S}_{\tilde{\tau}_n})$ has the same law as $(\tilde{\sigma}_n)$.
3. For $t \in [\tilde{\tau}_n, \tilde{\tau}_{n+1})$, then $Z_t = Z_{\tilde{\tau}_n}$.
4. Conditioned on the information known up to time $\tilde{\tau}_n$ for \tilde{S}_t , $t \in [0, \tilde{\tau}_n]$ and $\tilde{\tau}_m$, $m \leq n$, and also the value $\tilde{S}_{\tilde{\tau}_{n+1}}$, the law for the trajectories S_t , $t \in [\tilde{\tau}_n, \tilde{\tau}_{n+1}]$ (which includes the length $\tilde{\tau}_{n+1} - \tilde{\tau}_n$) agrees with the law for the original dynamics conditioned on knowing the values $S_{\tilde{\tau}_n}$ and $S_{\tilde{\tau}_{n+1}}$.

The marginal distribution for the first component S_t agrees with the original process and the times $\tilde{\tau}_n$ are independent mean-1 exponential random variables which are independent of S_t . Of course, the times $\tilde{\tau}_n$ are not independent of the process \tilde{S}_t , and we emphasize that the increment $\tilde{\tau}_{n+1} - \tilde{\tau}_n$ is not necessarily exponential given the state $\tilde{S}_{\tilde{\tau}_n}$. The process \tilde{S}_t is not Markovian due to the conditioning in (4), although the chain $(\tilde{S}_{\tilde{\tau}_n})$ is Markovian. By [17] we can construct a Markov process by including an extra component to the process: the triple $(S_t, Z_t, S_{\tau(t)}) \in \Sigma \times \{0, 1\} \times \Sigma$ is Markovian, where $\tau(t)$ is the first partition time to occur after time t . We refer to the statistics of the split process by $\tilde{\mathbb{E}}^{(\lambda)}$ and $\tilde{\mathbb{P}}^{(\lambda)}$ for expectations and probabilities, respectively. We will neglect the tilde from the symbol $\tilde{\tau}_n$ for the remainder of the text.

Now that we have defined the split process \tilde{S}_t , we can proceed to define the life cycles. Let R'_m , $m \geq 1$ be the time $\tau_{\tilde{n}_m}$ for $\tilde{n}_m = \min\{n \in \mathbb{N} \mid \sum_{r=0}^n \chi(Z_{\tau_r} = 1) = m\}$. The random variable R'_m is the m th partition time corresponding to a visit of the atom set $\Sigma \times 1$, and we set $R'_0 = 0$ by convention. We define R_m to be the partition time following R'_m . The m th life cycle is the time interval $[R_m, R_{m+1})$. Successive life cycle trajectories over $[R_{n-1}, R_n)$ and $[R_n, R_{n+1})$ are obviously not independent, since a.s. $S_{R_n^-} = S_{R_n}$. However, non-successive life cycles are independent. When considering the random variables $\int_{R_n}^{R_{n+1}} dr \frac{dV}{dx}(X_r)$, the correlations between successive terms can be removed by adding and subtracting certain resolvent terms as seen in the summand in the lemma below.

Let \tilde{N}_t be the number of R'_n to have occurred up to time t . Define $\tilde{\mathcal{F}}'_t$ to be the filtration containing all information for the partition times τ_n and the split process \tilde{S}_r before time R_{n+1} where $t \in [R'_n, R'_{n+1})$. Also define $\mathfrak{R}^{(\lambda)}$ as the reduced resolvent of the backward generator \mathcal{L}_λ corresponding to the master equation (1.1). The reduced resolvent formally satisfies $\mathfrak{R}^{(\lambda)} = \int_0^\infty dr e^{r\mathcal{L}_\lambda}$ on elements $g \in L^\infty(\Sigma)$ with $\Psi_{\infty,\lambda}(g) = 0$. Notice that the martingale defined in the lemma below resembles (1.10).

Lemma 2.1. *Let the process \tilde{M}_t be defined as*

$$\tilde{M}_t = \sum_{n=1}^{\tilde{N}_t} \left(\int_{R_n}^{R_{n+1}} dr \frac{dV}{dx}(X_r) - \left(\mathfrak{R}^{(\lambda)} \frac{dV}{dx} \right)(S_{R_n}) + \left(\mathfrak{R}^{(\lambda)} \frac{dV}{dx} \right)(S_{R_{n+1}}) \right).$$

The process \tilde{M}_t is a martingale with respect to the filtration $\tilde{\mathcal{F}}'_t$. Moreover, the predictable quadratic variation $\langle \tilde{M} \rangle_t$ has the form

$$\langle \tilde{M} \rangle_t = \sum_{n=1}^{\tilde{N}_t} \bar{v}_\lambda(S_{R_n}),$$

where $\bar{v}_\lambda : \Sigma \rightarrow \mathbb{R}^+$ is defined as

$$\begin{aligned} \bar{v}_\lambda(s) = 2\tilde{\mathbb{E}}_{\tilde{\delta}_s}^{(\lambda)} & \left[\int_0^{R_1} dr \frac{dV}{dx}(X_r) \left(\mathfrak{R}^{(\lambda)} \frac{dV}{dx} \right)(S_r) \right] + \int_{\Sigma} d\nu(s') \left(\left(\mathfrak{R}^{(\lambda)} \frac{dV}{dx} \right)(s') \right)^2 \\ & - \left(\left(\mathfrak{R}^{(\lambda)} \frac{dV}{dx} \right)(s) \right)^2 \end{aligned}$$

In the above, $\tilde{\delta}_s$ is the splitting of the δ -distribution at s (see (2.3)).

3 Convergence of a local time quantity

In this section, we work to prove Theorem 3.1 below. In the statement of the theorem, $L_t = \mathbf{u}^{-1} \int_0^t dr h(S_r)$, where $\mathbf{u} > 0$ and $h : \Sigma \rightarrow [0, 1]$ are defined as in Section 2. The importance of the process L_t is that it is close (on the relevant scale) to the bracket process $\langle \tilde{M} \rangle_t$ for the martingale \tilde{M}_t of Lemma 2.1.

Theorem 3.1. *Let \mathfrak{p}_t be the Ornstein-Uhlenbeck process and \mathfrak{l}_t be its local time at zero. As $\lambda \rightarrow 0$, there is convergence in law*

$$(\lambda^{\frac{1}{2}} P_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}}) \xrightarrow{\mathcal{L}} (\mathfrak{p}_t, \mathfrak{l}_t), \quad t \in [0, T],$$

where the convergence is with respect to the uniform metric. Moreover, for any t

$$\sup_{\lambda < 1} \mathbb{E}^{(\lambda)} [\lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}}] < \infty \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \mathbb{E}^{(\lambda)} [\lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}}] = \mathbb{E}[\mathfrak{l}_t].$$

We begin by making some remarks on the local time process \mathfrak{l} . Appendix A contains more information, although without proofs. Define $\tilde{\mathbf{B}}_t = \int_0^t dr \operatorname{sgn}(\mathfrak{p}_r) d\mathbf{B}'_r$, where \mathbf{B}' is the Brownian motion driving the Langevine equation (1.3) and $\operatorname{sgn} : \mathbb{R} \rightarrow \{\pm 1\}$ is the sign function. The Tanaka-Meyer formula yields the local time at zero for \mathfrak{p} as

$$\mathfrak{l}_t = |\mathfrak{p}_t| - |\mathfrak{p}_0| - \tilde{\mathbf{B}}_t + \frac{1}{2} \int_0^t dr |\mathfrak{p}_r|. \quad (3.1)$$

The above relation follows from the formal definition $\mathfrak{l}_t = \int_0^t dr \delta_0(\mathfrak{p}_r)$ and a formal application of the Ito formula for the function $|\cdot|$ of the process \mathfrak{p} which has differential $d\mathfrak{p}_t = -\frac{1}{2}\mathfrak{p}_t dt + d\mathbf{B}'_t$. In (3.1), \mathfrak{l} is the positive part of the drift for the diffusion process \mathfrak{p} .

Theorem 3.1 states that a rescaling of the process L_t converges in law to the local time \mathfrak{l}_t . Since $h(x, p)$ is compactly supported, it is not surprising that this quantity would be related to the local time when considered on the appropriate scale: $\lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}}$, $\lambda \ll 1$. The strategy in the proof resembles [5, Thm. 5.3] in which information related to the limiting behavior for the momentum process P_t is found through a study of the semimartingale decomposition of the square root energy process $\mathbf{Q}_t = (2H_t)^{\frac{1}{2}} = (P_t^2 + 2V(X_t))^{\frac{1}{2}}$. Since the potential $V(x)$ is bounded, we have that $\lambda^{\frac{1}{2}}|P_{\frac{t}{\lambda}}| \approx \lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}$. The advantage of working with a function of the Hamiltonian is that there is no drift between collisions. Let the processes \mathbf{M}_t , \mathbf{A}_t^+ , and $-\mathbf{A}_t^-$ be respectively the martingale, predictable increasing, and predictable decreasing parts in the semimartingale decomposition of the process \mathbf{Q}_t . The processes \mathbf{A}_t^\pm and the predictable quadratic variation $\langle \mathbf{M} \rangle_t$ of the martingale \mathbf{M}_t have the forms

$$\mathbf{A}_t^\pm = \int_0^t dr \mathcal{A}_\lambda^\pm(S_r) \quad \text{and} \quad \langle \mathbf{M} \rangle_t = \int_0^t dr \mathcal{V}_\lambda(S_r),$$

where $\mathcal{A}_\lambda^\pm, \mathcal{V}_\lambda$ are defined below.

Also, since L_t is difficult to work with directly, our strategy is to approximate it by \mathbf{A}_t^+ . Notice that we can rewrite the components in the semimartingale decomposition as

$$\mathbf{A}_t^+ = \mathbf{Q}_t - \mathbf{Q}_0 - \mathbf{M}_t + \mathbf{A}_t^-.$$

in analogy with the Tanaka-Meyer formula (3.1). We approach the term $\lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+$ through a study of the joint convergence of the terms

$$\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}} \xrightarrow{\mathcal{L}} |\mathbf{p}_t|, \quad \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}} \xrightarrow{\mathcal{L}} \tilde{\mathbf{B}}_t, \quad \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^- \xrightarrow{\mathcal{L}} \frac{1}{2} \int_0^t dr |\mathbf{p}_r|.$$

The next lemma gives a limiting procedure in which the trajectories for \mathfrak{l} and $\tilde{\mathbf{B}}$ in the Tanaka-Meyer formula (3.1) are determined by the trajectories for $|\mathbf{p}|$.

Lemma 3.2. *Let \mathbf{p}_t be the Ornstein-Uhlenbeck process. As $\epsilon \rightarrow 0$, the local time at zero \mathfrak{l} satisfies*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \mathfrak{l}_t - \frac{1}{2\epsilon} \int_0^t dr e^{-\frac{|\mathbf{p}_r|}{\epsilon}} \right| \right] = O(\epsilon^{\frac{1}{2}}).$$

Also, the Brownian motion $\tilde{\mathbf{B}}_t$ in the Tanaka-Meyer formula (3.1) satisfies

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \tilde{\mathbf{B}}_t - |\mathbf{p}_t| + |\mathbf{p}_0| - \epsilon e^{-\frac{|\mathbf{p}_t|}{\epsilon}} - \frac{1}{2} \int_0^t dr |\mathbf{p}_r| (1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}}) + \frac{1}{2\epsilon} \int_0^t dr e^{-\frac{|\mathbf{p}_r|}{\epsilon}} \right| \right] = O(\epsilon^{\frac{1}{2}}).$$

Proof. Define the martingale $\mathfrak{m}_{t,\epsilon} = \int_0^t d\tilde{\mathbf{B}}_r (1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}})$. The difference between $\mathfrak{m}_{t,\epsilon}$ and $\tilde{\mathbf{B}}_t$ tends to zero as $\epsilon \rightarrow 0$ in the norm $\mathbb{E} \left[\sup_{0 \leq t \leq T} |\cdot| \right]$, since

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq r \leq T} |\tilde{\mathbf{B}}_r - \mathfrak{m}_{r,\epsilon}| \right] &\leq \mathbb{E} \left[\sup_{0 \leq r \leq T} |\tilde{\mathbf{B}}_r - \mathfrak{m}_{r,\epsilon}|^2 \right]^{\frac{1}{2}} \leq 2\mathbb{E} \left[|\tilde{\mathbf{B}}_T - \mathfrak{m}_{T,\epsilon}|^2 \right]^{\frac{1}{2}} \\ &= 2\mathbb{E} \left[\int_0^T dr e^{-2\frac{|\mathbf{p}_r|}{\epsilon}} \right]^{\frac{1}{2}} = 2 \left(\int_0^T dr \mathbb{E} \left[e^{-\frac{|\mathbf{p}_r|}{\epsilon}} \right] \right)^{\frac{1}{2}} \leq 2 \left(\int_0^T dr \mathbb{E}_0 \left[e^{-2\frac{|\mathbf{p}_r|}{\epsilon}} \right] \right)^{\frac{1}{2}} \\ &= 2 \left(\int_0^T dr \int_{\mathbb{R}} dq \frac{e^{-2\frac{1}{2\omega_r} q^2 - 2\frac{|q|}{\epsilon}}}{(2\pi\omega_r)^{\frac{1}{2}}} \right)^{\frac{1}{2}} = O(\epsilon^{\frac{1}{2}}), \end{aligned} \tag{3.2}$$

where $\omega_t = 1 - e^{-t}$. The first inequality is Jensen's, the second is Doob's, and the first equality uses that $e^{-2\frac{|\mathbf{p}_r|}{\epsilon}}$ is the quadratic variation of the martingale $\tilde{\mathbf{B}}_r - \mathbf{m}_{r,\epsilon}$. The third inequality uses that $\mathbb{E}[e^{-2\frac{|\mathbf{p}_r|}{\epsilon}}]$ is smallest when \mathbf{p}_0 is initially zero. The third equality holds since $\frac{e^{-\frac{1}{2\omega_t}q^2}}{(2\pi\omega_t)^{\frac{1}{2}}}$ is the density for \mathbf{p}_t starting with $\mathbf{p}_0 = 0$.

Moreover, $\mathbf{m}_{t,\epsilon}$ can be rewritten

$$\begin{aligned}\mathbf{m}_{t,\epsilon} = \int_0^t d\tilde{\mathbf{B}}_r (1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}}) &= \int_0^t (d|\mathbf{p}_t| + \frac{1}{2}dr|\mathbf{p}_r|)(1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}}) \\ &= |\mathbf{p}_t| - |\mathbf{p}_0| + \epsilon e^{-\frac{|\mathbf{p}_t|}{\epsilon}} + \frac{1}{2} \int_0^t dr |\mathbf{p}_r| (1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}}) - \frac{1}{2\epsilon} \int_0^t dr e^{-\frac{|\mathbf{p}_r|}{\epsilon}}.\end{aligned}$$

The second equality follows by the substitution $d\tilde{\mathbf{B}}_t = d|\mathbf{p}_t| - \frac{1}{2}dt|\mathbf{p}_t| - d\mathbf{l}_t$ (from the Tanaka-Meyer formula (3.1)) and since $d\mathbf{l}_t$ multiplied by $(1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}})$ is zero. The chain rule and the fact that $(d|\mathbf{p}_r|)^2 = dr$ give the third equality. From the convergence (3.2), it follows that the right side converges to $\tilde{\mathbf{B}}$ in the norm $\|\cdot\|_{\mathfrak{s}} = \mathbb{E}[\sup_{0 \leq t \leq T} |\cdot|]$.

As $\epsilon \rightarrow 0$,

$$\left\| \epsilon e^{-\frac{|\mathbf{p}_t|}{\epsilon}} \right\|_{\mathfrak{s}} = O(\epsilon) \quad \text{and} \quad \left\| \int_0^t dr |\mathbf{p}_r| e^{-\frac{|\mathbf{p}_r|}{\epsilon}} \right\|_{\mathfrak{s}} = O(\epsilon),$$

where the later term follows by the same argument as in the right side of (3.2). In conclusion,

$$\tilde{\mathbf{B}}_t = |\mathbf{p}_t| - |\mathbf{p}_0| + \frac{1}{2} \int_0^t dr |\mathbf{p}_r| - \int_0^t dr e^{-\frac{|\mathbf{p}_r|}{\epsilon}} + O(\epsilon^{\frac{1}{2}}),$$

where $O(\epsilon^{\frac{1}{2}})$ refers to the norm $\|\cdot\|_{\mathfrak{s}}$. By the Tanaka-Meyer formula

$$\mathbf{l}_t = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t dr e^{-\frac{|\mathbf{p}_r|}{\epsilon}},$$

where the error in the limit is $O(\epsilon^{\frac{1}{2}})$ in $\|\cdot\|_{\mathfrak{s}}$. □

Before proceeding to the proof of Theorem 3.1, we must recall some of the notation and a few of the results from [6]. Define the functions $\mathcal{A}_\lambda, \mathcal{V}_\lambda, \mathcal{K}_{\lambda,n} : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\begin{aligned}\mathcal{A}_\lambda(x, p) &= \int_{\mathbb{R}} dp' (2^{\frac{1}{2}} H(x, p')^{\frac{1}{2}} - 2^{\frac{1}{2}} H(x, p)^{\frac{1}{2}}) \mathcal{J}_\lambda(p, p'), \\ \mathcal{V}_\lambda(x, p) &= \int_{\mathbb{R}} dp' \left(2^{\frac{1}{2}} H(x, p')^{\frac{1}{2}} - 2^{\frac{1}{2}} H(x, p)^{\frac{1}{2}} - \frac{\mathcal{A}_\lambda(x, p)}{\mathcal{E}_\lambda(p)} \right)^2 \mathcal{J}_\lambda(p, p'), \\ \mathcal{K}_{\lambda,n}(x, p) &= \int_{\mathbb{R}} dp' |H(x, p')^{\frac{1}{2}} - H(x, p)^{\frac{1}{2}}|^n \mathcal{J}_\lambda(p, p'),\end{aligned}$$

where $\mathcal{E}_\lambda(p) = \int_{\mathbb{R}} dp' \mathcal{J}_\lambda(p, p')$ are the escape rates. We define $\mathcal{A}_\lambda^\pm(s) = \max(\pm\mathcal{A}_\lambda(s), 0)$ to be the positive and negative parts of \mathcal{A}_λ . Proposition 3.3 is a combination of Propositions 2.1, 3.1, and 4.15 of [6] and contains some basic inequalities regarding the functions $\mathcal{A}_\lambda^\pm, \mathcal{V}_\lambda$, and $\mathcal{K}_{\lambda,n}$.

Proposition 3.3. *There are constants $c, C_n > 0$ such that for λ small enough,*

1. *For all $(x, p) \in \Sigma$, $\mathcal{K}_{\lambda,n}(x, p) \leq C_n(1 + \lambda|p|)^{n+1}$.*

2. For all $(x, p) \in \Sigma$, $\mathcal{V}_\lambda(x, p) \leq C(1 + \lambda|p|)$.
 3. For all $(x, p) \in \Sigma$, $\mathcal{A}_\lambda^+(x, p) \leq \frac{C}{1+p^2}$.
 4. For $\lambda^{-\frac{3}{8}} \leq |p| \leq \lambda^{-\frac{3}{4}}$,
- $$\left| \mathcal{A}_\lambda^-(x, p) - \frac{1}{2}\lambda|p| \right| \leq C\lambda^{\frac{5}{4}}|p|, \quad \left| \mathcal{V}_\lambda(x, p) - 1 \right| \leq C\lambda^{\frac{1}{2}}, \quad \text{and} \quad \left| 2\mathcal{K}_{\lambda,2}(x, p) - 1 \right| \leq C\lambda^{\frac{1}{2}}.$$
5. For all $(x, p) \in \Sigma$, $\mathcal{A}_\lambda^-(x, p) \leq |\mathcal{D}_\lambda(p)|$. In particular, for $|p| \leq \lambda^{-1}$, one has $\mathcal{A}_\lambda^-(x, p) \leq C\lambda|p|$.
 6. For all $(x, p) \in \Sigma$, $\left| \frac{\mathcal{A}_\lambda(x, p)}{\mathcal{E}_\lambda(p)} + \frac{2\lambda|p|}{1+\lambda} \right| \leq C$.
 7. $\mathcal{E}_\lambda(p) \leq \frac{1}{8(\lambda+1)} (1 + C \min(\lambda|p|, \lambda^2 p^2))$ and $\lambda|p| \leq C\mathcal{E}_\lambda(p)$.

Lemmas 3.4 and 3.5 below are both from [6, Sec. 2], and they characterize the typical energy behavior over the time interval $[0, \frac{T}{\lambda}]$ for $\lambda \ll 1$. In particular, Lemma 3.4 states that the energy $H(X_t, P_t) = \frac{1}{2}P_t^2 + V(X_t)$ does not typically go above the scale λ^{-1} , and Lemma 3.5 states that the energy typically does not spend much time smaller than $\lambda^{-\varrho}$ for any $0 \leq \varrho < 1$.

Lemma 3.4. For any $n \in \mathbb{N}$, there exists a $C > 0$ such that

$$\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq r \leq \frac{T}{\lambda}} (H_r)^{\frac{n}{2}} \right] \leq C \left(\frac{T}{\lambda} \right)^{\frac{n}{2}}$$

for all $T > 0$ and $\lambda < 1$.

Lemma 3.5. Define $\mathbf{T}_t = \lambda \int_0^t dr \chi(H_r \leq \epsilon \lambda^{-\varrho})$ for $0 \leq \varrho \leq 1$. For any fixed $T > 0$, there is a $C > 0$ such that for small enough λ and all $\epsilon \geq \lambda^\varrho$,

$$\mathbb{E}^{(\lambda)} [\mathbf{T}_{\frac{T}{\lambda}}] \leq C \epsilon^{\frac{1}{2}} \lambda^{\frac{1-\varrho}{2}}.$$

[Proof of Theorem 3.1]

By [6, Thm. 1.2], the process $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$ converges in law to the Ornstein-Uhlenbeck process \mathbf{p} with respect to the uniform metric. It is sufficient for us to show that $(|\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}|, \lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}})$ converges in law to the pair $(|\mathbf{p}|, \mathbf{l})$. Our approach will be to approximate the pair $(|\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}|, \lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}})$ by the pair $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+)$ in Part (i) below, and then to apply an argument based on the Tanaka-Meyer formula to analyze $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+)$ in Part (ii). All convergences in law in this proof are with respect to the uniform metric.

(i). Showing that $|\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}|$ is close to $\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}$ is easy, since

$$|(p^2 + 2V(x))^{\frac{1}{2}} - |p|| \leq (2 \sup_x V(x))^{\frac{1}{2}} \quad \text{and thus} \quad |\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}} - \lambda^{\frac{1}{2}}|P_{\frac{t}{\lambda}}|| \leq \lambda^{\frac{1}{2}} (2 \sup_x V(x))^{\frac{1}{2}}.$$

By [6, Lem. 4.17], $\lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}}$ approaches $\lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+$ in the sense that for $\lambda \ll 1$,

$$\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}} - \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+ \right| \right] = O(\lambda^{\frac{1}{2}}).$$

Also by [6, Lem. 4.17], the expectation $\mathbb{E}^{(\lambda)}[\lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}}]$ is uniformly bounded for $\lambda < 1$. A consequence of Part (ii) will be that $\lambda^{\frac{1}{2}}L_{\frac{t}{\lambda}}$ converges in law to \mathbf{l} as $\lambda \rightarrow 0$. This implies convergence of the first moment.

(ii). The process $\lambda^{\frac{1}{2}}|P_{\frac{t}{\lambda}}|$ converges in law to $|\mathbf{p}|$, since $|\cdot|$ is a continuous map on functions in $L^\infty([0, T])$ with respect to the supremum norm and $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$ converges in law to \mathbf{p} by [6, Thm. 1.2]. With Part (i), it follows that $\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}$ converges in law to $|\mathbf{p}|$. Our main work is to incorporate the component $\lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+$ for the convergence in law of the pair $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+)$.

For the process \mathbf{A}_t^+ , we may write

$$\mathbf{A}_t^+ = \mathbf{Q}_t - \mathbf{Q}_0 - \mathbf{M}_t + \mathbf{A}_t^- . \quad (3.3)$$

Now, we will begin the analysis of $\lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+$ through a study of the terms on the right side of the above equation. By our assumptions on the initial distribution μ for (X_0, P_0) , the random variable $\lambda^{\frac{1}{2}}\mathbf{Q}_0$ converges to zero in probability. We will show that there is convergence in law

$$\mathbf{Y}_t^{(\lambda)} = (\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^-) \xrightarrow{\mathcal{D}} \left(|\mathbf{p}_t|, \tilde{\mathbf{B}}_t, \frac{1}{2} \int_0^t dr |\mathbf{p}_r| \right), \quad (3.4)$$

where $\tilde{\mathbf{B}}$ is the copy of Brownian motion in the Tanaka-Meyer formula (3.1). With the identities (3.1) and (3.3), the above convergence implies that $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^+)$ converges in law to $(|\mathbf{p}|, \mathbf{l})$. To prove the convergence (3.4), we will first show that $\lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^-$ can be approximated by $\frac{1}{2} \int_0^t dr \lambda^{\frac{1}{2}}\mathbf{Q}_r$ (see (I) below). It is then enough to show functional convergence of the pair $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}})$, since the map sends $q \in L^\infty([0, T])$ to the element $\frac{1}{2} \int_0^t dr q_r \in L^\infty([0, T])$ is continuous with respect to the supremum norm. A similar idea applies in the proof of the convergence in law of $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}})$. It is clear from the statement of Lemma 3.2 that the trajectories for $|\mathbf{p}|$ determine the trajectories for $\tilde{\mathbf{B}}$, and the same relation emerges between $\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}$ and $\lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}}$ in the limit $\lambda \rightarrow 0$. The main idea of the proof is to reduce everything to the functional convergence of $\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}$ to the absolute value of the Ornstein-Uhlenbeck process $|\mathbf{p}|$, which we know to occur by the observation following (ii) above.

The analysis below will be split into the proof of statements (I)-(III) below. The proofs of (II) and (III) work toward the convergence of the pair $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}})$.

(I). There is $C > 0$ such that for all $\lambda \leq 1$,

$$\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}}\mathbf{A}_{\frac{t}{\lambda}}^- - \frac{1}{2} \int_0^t dr \lambda^{\frac{1}{2}}\mathbf{Q}_r \right| \right] \leq C\lambda^{\frac{1}{8}}.$$

(II). The martingales $\mathbf{m}_{t,\epsilon}^{(\lambda)}$ defined as

$$\mathbf{m}_{t,\epsilon}^{(\lambda)} = \lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{M}_r (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r-}})$$

are close to $\lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}}$ for small λ and ϵ in the sense

$$\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}} - \mathbf{m}_{t,\epsilon}^{(\lambda)} \right|^2 \right] \leq C(\epsilon \vee \lambda)^{\frac{1}{2}} \quad (3.5)$$

for some C and all $\lambda, \epsilon < 1$.

(III). For each fixed ϵ , there is convergence in law as $\lambda \rightarrow 0$

$$(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \mathbf{m}_{t,\epsilon}^{(\lambda)}) \xrightarrow{\mathcal{L}} (|\mathbf{p}_t|, \mathbf{m}_{t,\epsilon}),$$

$$\text{for } \mathbf{m}_{t,\epsilon} = \int_0^t d\tilde{\mathbf{B}}_r \left(1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}} \right).$$

The $\epsilon \vee \lambda$ on the right side of the inequality (3.5) can be replaced with ϵ by having a slightly more refined version of Lemma 3.5 which we do not require here. By combining the results (II) and (III) with Lemma 3.2, which gives the convergence of $(\mathbf{p}, \mathbf{m}_{\cdot,\epsilon})$ to $(\mathbf{p}, \tilde{\mathbf{B}})$ in the norm $\|\cdot\|_{\mathfrak{s}} = \mathbb{E}[\sup_{0 \leq t \leq T} |\cdot|]$ as $\epsilon \rightarrow 0$, then a standard argument which we sketch below shows that $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{\cdot}{\lambda}})$ converges in law to $(\mathbf{p}, \tilde{\mathbf{B}})$. These statements can be summarized by the marked arrows in the diagram below

$$\begin{array}{ccc} (\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \mathbf{m}_{t,\epsilon}^{(\lambda)}) & \xrightarrow{\mathcal{L}} & (\mathbf{p}_t, \mathbf{m}_{t,\epsilon}) \\ \downarrow \|\cdot\|_{\mathfrak{s}} & & \downarrow \|\cdot\|_{\mathfrak{s}} \\ (\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{t}{\lambda}}) & \Longrightarrow & (\mathbf{p}_t, \tilde{\mathbf{B}}_t) \end{array},$$

where the convergence on the right side of the diagram is by Lemma 3.2, the top of the diagram is by (III), and the converge on the left side of the diagram is from (II) and requires both ϵ and λ to be small. Let us sketch the proof of the convergence in law at the bottom line of the diagram. By [23, Cor. IV.2.9], it is enough to show the convergence as $\lambda \rightarrow 0$ of

$$|\mathbb{E}^{(\lambda)} [F(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{\cdot}{\lambda}})] - \mathbb{E}^{(\lambda)} [F(\mathbf{p}, \tilde{\mathbf{B}})]| \quad (3.6)$$

to zero for functionals $F : L^\infty([0, T], \mathbb{R}^2) \rightarrow \mathbb{R}$ which are bounded and uniformly continuous with respect to the supremum norm. By the triangle inequality (3.6) is smaller than

$$\begin{aligned} & |\mathbb{E}^{(\lambda)} [F(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{2}}\mathbf{M}_{\frac{\cdot}{\lambda}})] - \mathbb{E}^{(\lambda)} [F(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{\cdot}{\lambda}}, \mathbf{m}_{\cdot,\epsilon}^{(\lambda)})]| + |\mathbb{E}^{(\lambda)} [F(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{\cdot}{\lambda}}, \mathbf{m}_{\cdot,\epsilon}^{(\lambda)})] - \mathbb{E}^{(\lambda)} [F(\mathbf{p}, \mathbf{m}_{\cdot,\epsilon})]| \\ & \quad + |\mathbb{E}^{(\lambda)} [F(\mathbf{p}, \mathbf{m}_{\cdot,\epsilon})] - \mathbb{E}^{(\lambda)} [F(\mathbf{p}, \tilde{\mathbf{B}})]|. \end{aligned} \quad (3.7)$$

Since F is bounded and uniformly continuous, we can choose $\epsilon \vee \lambda$ and ϵ to make both the first and third terms small by (III) and Lemma 3.2, respectively. We can then choose $\lambda \in (0, \epsilon]$ to make the second term arbitrarily small by the convergence (II).

Next, we prove statements (I)-(III). The definition of constants $C_n, C'_n > 0$, $n \in \mathbb{N}$ will reset in different parts of the analysis.

(I). By the remark (ii), it is sufficient to bound the difference between $\lambda^{\frac{1}{2}} \mathbf{A}_{\frac{t}{\lambda}}^-$ and $\frac{1}{2} \int_0^t dr |\lambda^{\frac{1}{2}} P_{\frac{r}{\lambda}}|$ for small λ . Conditioned on the event that $\lambda^{-\frac{3}{4}}$ for $t \in [0, \frac{T}{\lambda}]$, then

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \mathbf{A}_{\frac{t}{\lambda}}^- - \frac{1}{2} \int_0^t dr |\lambda^{\frac{1}{2}} P_{\frac{r}{\lambda}}| \right| \\ & \leq \lambda^{\frac{1}{8}} C_1 \int_0^T dr \chi(|P_{\frac{r}{\lambda}}| \leq \lambda^{-\frac{3}{8}}) + \lambda^{\frac{1}{2}} \int_0^T dr \chi(|P_{\frac{r}{\lambda}}| \geq \lambda^{-\frac{3}{8}}) \left| \lambda^{-1} \mathcal{A}_\lambda^-(X_{\frac{r}{\lambda}}, P_{\frac{r}{\lambda}}) - \frac{1}{2} |P_{\frac{r}{\lambda}}| \right| \\ & \leq C_1 T \lambda^{\frac{1}{8}} + C_2 T \lambda^{\frac{3}{4}} \sup_{0 \leq r \leq \frac{T}{\lambda}} |P_r|, \end{aligned}$$

where $C_1 := \frac{1}{2} + \sup_{|p| \leq \lambda^{-\frac{3}{8}}} \lambda^{-\frac{5}{8}} \mathcal{A}_\lambda^-(x, p)$, and C_1 is finite by Part (5) of Proposition 3.3. The $C_2 > 0$ in the second inequality is from Part (4) of Proposition 3.3.

The above implies the first inequality below,

$$\begin{aligned} \mathbb{E}^{(\lambda)} \left[\chi \left(\sup_{0 \leq r \leq \frac{T}{\lambda}} |P_r| \leq \lambda^{-\frac{3}{4}} \right) \sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \mathbf{A}_{\frac{t}{\lambda}}^- - \frac{1}{2} \int_0^t dr |\lambda^{\frac{1}{2}} P_{\frac{r}{\lambda}}| \right| \right] & \leq C_1 T \lambda^{\frac{1}{8}} + C_2 \lambda^{\frac{3}{4}} \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq r \leq \frac{T}{\lambda}} |P_r| \right] \\ & \leq C_1 T \lambda^{\frac{1}{8}} + C_2 2^{-\frac{1}{2}} T \lambda^{\frac{3}{4}} \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq r \leq \frac{T}{\lambda}} \mathbf{Q}_r \right] \leq C_1 T \lambda^{\frac{1}{8}} + C'_2 T^{\frac{3}{2}} \lambda^{\frac{1}{4}}, \end{aligned}$$

where the second and third inequalities follows from $P_r^2 \leq 2H_r$ and by Lemma 3.4, respectively. Moreover, for the event $\sup_{0 \leq r \leq \frac{T}{\lambda}} |P_r| > \lambda^{-\frac{3}{4}}$, then

$$\begin{aligned} & \mathbb{E}^{(\lambda)} \left[\chi \left(\sup_{0 \leq r \leq \frac{T}{\lambda}} |P_r| > \lambda^{-\frac{3}{4}} \right) \sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \mathbf{A}_{\frac{t}{\lambda}}^- - \frac{1}{2} \int_0^t dr |\lambda^{\frac{1}{2}} P_{\frac{r}{\lambda}}| \right| \right] \\ & \leq \mathbb{P}^{(\lambda)} \left[\sup_{0 \leq r \leq \frac{T}{\lambda}} |P_r| > \lambda^{-\frac{3}{4}} \right]^{\frac{1}{2}} \mathbb{E}^{(\lambda)} \left[\left| \int_0^T dr \left(|\lambda^{\frac{1}{2}} P_{\frac{r}{\lambda}}| + \lambda^{\frac{1}{2}} \mathcal{A}_\lambda^-(X_{\frac{r}{\lambda}}, P_{\frac{r}{\lambda}}) \right) \right|^2 \right]^{\frac{1}{2}} \\ & \leq C'_1 \lambda^{\frac{1}{4}} T^{\frac{1}{2}} \mathbb{E}^{(\lambda)} \left[\left(\sup_{0 \leq r \leq \frac{T}{\lambda}} \lambda^{\frac{1}{2}} |P_r| \right)^2 \right]^{\frac{1}{2}} \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq r \leq \frac{T}{\lambda}} (\lambda^{\frac{1}{2}} + \lambda^{\frac{1}{2}} |P_r| + \lambda^{\frac{5}{2}} |P_r|^2)^2 \right]^{\frac{1}{2}} = O(\lambda^{\frac{1}{4}}). \end{aligned}$$

The first inequality is Cauchy-Schwarz, and the second is Chebyshev's for the first term. For the second term in the second inequality, Part (6) and (7) of Proposition 3.3 state that there are $C_1, C'_1 > 0$ such that

$$|p| + \mathcal{A}_\lambda^-(x, p) \leq |p| + 4\lambda|p|\mathcal{E}_\lambda(p) + C_1 \mathcal{E}_\lambda(p) \leq C'_1 (1 + |p| + \lambda^2 |p|^2).$$

The expectations on the last line above are finite by Lemma 3.4, since $|P_r| \leq (2H_r)^{\frac{1}{2}}$.

(II). The difference between $\lambda^{\frac{1}{2}} \mathbf{M}_{\frac{t}{\lambda}}$ and $\mathbf{m}_{t,\epsilon}^{(\lambda)}$ can be bounded by

$$\begin{aligned} \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \mathbf{M}_{\frac{t}{\lambda}} - \mathbf{m}_{t,\epsilon}^{(\lambda)} \right|^2 \right] & \leq 4 \mathbb{E}^{(\lambda)} \left[\left| \lambda^{\frac{1}{2}} \mathbf{M}_{\frac{T}{\lambda}} - \mathbf{m}_{T,\epsilon}^{(\lambda)} \right|^2 \right] = 4\lambda \mathbb{E}^{(\lambda)} \left[\left| \int_0^{\frac{T}{\lambda}} d\mathbf{M}_r e^{-\epsilon^{-1} \lambda^{\frac{1}{2}} \mathbf{Q}_{r-}} \right|^2 \right] \\ & = 4 \mathbb{E}^{(\lambda)} \left[\lambda \int_0^{\frac{T}{\lambda}} dr \mathcal{V}_\lambda(X_r, P_r) e^{-2\epsilon^{-1} \lambda^{\frac{1}{2}} \mathbf{Q}_{r-}} \right]. \end{aligned}$$

The first inequality is Doob's, and the second equality uses that $\frac{d}{dt}\langle \mathbf{M} \rangle_t = \mathcal{V}_\lambda(X_t, P_t)$. For $\epsilon \in [\lambda, 1]$, the right side is smaller than

$$\begin{aligned} \mathbb{E}^{(\lambda)} \left[\lambda \int_0^{\frac{T}{\lambda}} dr \mathcal{V}_\lambda(X_r, P_r) e^{-2\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} \right] &\leq C_1 \mathbb{E}^{(\lambda)} \left[\mathbf{T}_{\frac{T}{\lambda}} \right] + T \sup_{|p| > \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}}} \mathcal{V}_\lambda(x, p) e^{-2^{\frac{3}{2}}\epsilon^{-1}\lambda^{\frac{1}{2}}H^{\frac{1}{2}}(x, p)} \\ &\leq C_1 \mathbb{E}^{(\lambda)} \left[\mathbf{T}_{\frac{T}{\lambda}} \right] + C_2 T \sup_{|p| > \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}}} (1 + \lambda|p|) e^{-2\epsilon^{-1}\lambda^{\frac{1}{2}}|p|} \\ &\leq C'_1 (\epsilon \vee \lambda)^{\frac{1}{2}} + 2C_2 T e^{-2\epsilon^{-\frac{1}{2}}} = O(\epsilon^{\frac{1}{2}} \vee \lambda^{\frac{1}{2}}), \end{aligned}$$

where $C_1 := \sup_{\lambda < 1} \sup_{|p| \leq \lambda^{-1}} \mathcal{V}_\lambda(x, p)$ and $\mathbf{T}_t = \lambda \int_0^t dr \chi(H_r \leq \epsilon\lambda^{-1})$. The value C_1 is finite by Part (2) of Proposition 3.3. The second inequality uses Part (2) of Proposition 3.3 again, and we use that $|p| \leq 2^{\frac{1}{2}}H^{\frac{1}{2}}(x, p)$ in the exponent. The C'_1 in the third inequality is from Lemma 3.5.

(III). We will show that $\mathbf{m}_{t,\epsilon}^{(\lambda)}$ becomes close in the norm $\|\cdot\|_s$ to $F_t(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}})$ as $\lambda \rightarrow 0$ for a function $F : L^\infty([0, T]) \rightarrow L^\infty([0, T])$ which is continuous with respect to the supremum norm. The convergence in law of the pair $(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, F_t(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}))$ is then determined by the convergence of the first component.

For $q \in L^\infty([0, T])$, we define $F_t(q)$ as

$$F_t(q) = q_t + \epsilon e^{-\epsilon^{-1}q_t} + \frac{1}{2} \int_0^t dr q_r (1 - e^{-\epsilon^{-1}q_r}) - \frac{1}{2\epsilon} \int_0^t dr e^{-\epsilon^{-1}q_r}. \quad (3.8)$$

$F : L^\infty([0, T])$ is Lipschitz continuous with respect the supremum norm for a constant that scales as $\propto \epsilon^{-1}$ for small ϵ . Let $\mathbf{m}_{t,\epsilon}^{(\lambda),\prime} = F_t(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}})$. Notice that

$$\begin{aligned} F_t(|\mathbf{p}|) &= |\mathbf{p}_t| + \epsilon e^{-\frac{|\mathbf{p}_t|}{\epsilon}} + \frac{1}{2} \int_0^t dr |\mathbf{p}_r| (1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}}) - \frac{1}{2\epsilon} \int_0^t dr e^{-\frac{|\mathbf{p}_r|}{\epsilon}} \\ &= \int_0^t d\tilde{\mathbf{B}}_r (1 - e^{-\frac{|\mathbf{p}_r|}{\epsilon}}) = \mathbf{m}_{t,\epsilon} \end{aligned}$$

where the second equality is from $d\tilde{\mathbf{B}}_t = d|\mathbf{p}_t| + \frac{1}{2}|\mathbf{p}_t|dt - d\mathbf{l}_t$, the chain rule, and that $(d|\mathbf{p}_t|)^2 = dt$. By (i) and the convergence of $\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}$ to \mathbf{p}_t by [6, Thm 1.2], there is convergence in law as $\lambda \rightarrow 0$,

$$(\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}, \mathbf{m}_{t,\epsilon}^{(\lambda),\prime}) \xrightarrow{\mathcal{L}} (|\mathbf{p}_t|, \mathbf{m}_{t,\epsilon}).$$

The remainder of the proof will focus on showing that the difference between $\mathbf{m}_{t,\epsilon}^{(\lambda)}$ and $\mathbf{m}_{t,\epsilon}^{(\lambda),\prime}$ converges to zero in the norm $\|\cdot\|_s$ as $\lambda \rightarrow 0$. More precisely, we show that $\|\mathbf{m}_{t,\epsilon}^{(\lambda)} - \mathbf{m}_{t,\epsilon}^{(\lambda),\prime}\|_s$ is $O(\lambda^{\frac{1}{8}})$ for small λ .

By substituting $d\mathbf{M}_r = d\mathbf{Q}_r - d\mathbf{A}_r^+ + d\mathbf{A}_r^-$, the martingale $\mathbf{m}_{t,\epsilon}^{(\lambda)}$ can be written as

$$\mathbf{m}_{t,\epsilon}^{(\lambda)} = \lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} (d\mathbf{Q}_r - d\mathbf{A}_r^+ + d\mathbf{A}_r^-) (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}}).$$

It is sufficient to show that

$$-\lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{A}_r^+ (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}}) \longrightarrow 0, \quad (3.9)$$

$$\lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{A}_r^- (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}}) - \frac{1}{2} \int_0^t dr \lambda^{\frac{1}{2}} \mathbf{Q}_{\frac{r}{\lambda}} (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{r}{\lambda}}}) \longrightarrow 0, \quad (3.10)$$

$$\lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{Q}_r (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}}) - \lambda^{\frac{1}{2}} \mathbf{Q}_{\frac{t}{\lambda}} - \epsilon e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}} + \frac{1}{2\epsilon} \int_0^t dr e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{r}{\lambda}}} \longrightarrow 0, \quad (3.11)$$

since the expressions sum up to $\mathbf{m}_{t,\epsilon}^{(\lambda)} - \mathbf{m}_{t,\epsilon}^{(\lambda),\prime}$.

Since $d\mathbf{A}_t^+ = dt \mathcal{A}_\lambda^+(X_t, P_t)$, the value (3.9) is bounded by

$$\begin{aligned} & \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{A}_r^+ (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}}) \right| \right] = \mathbb{E}^{(\lambda)} \left[\lambda \int_0^{\frac{T}{\lambda}} dr \mathcal{A}_\lambda^+(X_r, P_r) (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_r}) \right] \\ & \leq C \lambda \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr \frac{1}{1 + |P_r|^2} (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_r}) \right] \leq C \mathbb{E}^{(\lambda)} [\mathbf{T}_{\frac{T}{\lambda}}] + CT \sup_{|p| > \epsilon^{\frac{1}{2}} \lambda^{-\frac{1}{2}}} \frac{e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}|p|}}{1 + p^2} = O(\epsilon^{\frac{1}{2}}), \end{aligned}$$

where \mathbf{T}_t is defined as above. The first inequality is from Part (3) of Proposition 3.3, and the second inequality is similar to the analysis in Part (I). For the convergence (3.10), $d\mathbf{A}_t^- = dt \mathcal{A}_\lambda^-(X_t, P_t)$ and

$$\begin{aligned} & \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{A}_r^- (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}}) - \lambda^{\frac{1}{2}} \frac{1}{2} \int_0^t dr \mathbf{Q}_{\frac{r}{\lambda}} (1 - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{r}{\lambda}}}) \right| \right] \\ & \leq \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \int_0^t dr \left| \lambda^{-\frac{1}{2}} \mathcal{A}_\lambda^-(X_{\frac{r}{\lambda}}, P_{\frac{r}{\lambda}}) - \frac{1}{2} \lambda^{\frac{1}{2}} \mathbf{Q}_{\frac{r}{\lambda}} \right| \right]. \end{aligned}$$

By adding and subtracting $\frac{1}{2}\lambda^{\frac{1}{2}}|P_{\frac{r}{\lambda}}|$ in the integrand and applying the triangle inequality, we are left with terms

$$\left| \lambda^{-\frac{1}{2}} \mathcal{A}_\lambda^-(X_{\frac{r}{\lambda}}, P_{\frac{r}{\lambda}}) - \frac{1}{2} \lambda^{\frac{1}{2}} |P_{\frac{r}{\lambda}}| \right| \quad \text{and} \quad \left| \frac{1}{2} \lambda^{\frac{1}{2}} |P_{\frac{r}{\lambda}}| - \frac{1}{2} \lambda^{\frac{1}{2}} \mathbf{Q}_{\frac{r}{\lambda}} \right|,$$

which are bounded by the analysis in Part (II) and at the beginning of Part (i), respectively.

The convergence (3.11) requires more work. The terms $\lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{Q}_r$ and $\lambda^{\frac{1}{2}} \mathbf{Q}_{\frac{t}{\lambda}} - \lambda^{\frac{1}{2}} \mathbf{Q}_0$ are equal, and $\lambda^{\frac{1}{2}} \mathbf{Q}_0$ is small, so we must bound

$$\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \epsilon e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}} - \lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{Q}_r e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} - \frac{1}{2\epsilon} \int_0^{\frac{t}{\lambda}} dr e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{r}{\lambda}}} \right| \right]. \quad (3.12)$$

The difference would be zero by the Ito chain rule if $\lambda^{\frac{1}{2}} \mathbf{Q}_{\frac{t}{\lambda}}$ were replaced by $|\mathbf{p}_r|$, and the norm of the difference is essentially a measure of how close the chain rule is to holding. We start with a Taylor expansion around each collision time t_n . Let $\Delta \mathbf{Q}_r = \mathbf{Q}_r - \mathbf{Q}_{r^-}$, then $\epsilon e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}}$

can be written as

$$\begin{aligned}
\epsilon e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{\frac{t}{\lambda}}} - \epsilon e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_0} &= \epsilon \sum_{n=1}^{\mathcal{N}_{\frac{t}{\lambda}}} \left(e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{t_n}} - e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{t_n^-}} \right) = -\lambda^{\frac{1}{2}} \sum_{n=1}^{\mathcal{N}_{\frac{t}{\lambda}}} \Delta \mathbf{Q}_{t_n} e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{t_n^-}} \\
&\quad + \frac{\lambda}{2\epsilon} \sum_{n=1}^{\mathcal{N}_{\frac{t}{\lambda}}} (\Delta \mathbf{Q}_{t_n})^2 e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{t_n^-}} - \frac{\lambda^{\frac{3}{2}}}{\epsilon^2} \sum_{n=1}^{\mathcal{N}_{\frac{t}{\lambda}}} \int_0^{\Delta \mathbf{Q}_{t_n}} (\Delta \mathbf{Q}_{t_n} - w)^2 e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}(\mathbf{Q}_{t_n^-} + w)} \\
&= \lambda^{\frac{1}{2}} \int_0^{\frac{t}{\lambda}} d\mathbf{Q}_r e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} + \frac{\lambda}{2\epsilon} \int_0^{\frac{t}{\lambda}} (d\mathbf{Q}_r)^2 e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} + \mathbf{R}_{\lambda,\epsilon,t},
\end{aligned}$$

where \mathcal{N}_t is the number of collisions up to time t , and $\mathbf{R}_{\lambda,\epsilon,t}$ denotes the third term between the two equalities. By the triangle inequality, the expectation (3.12) is smaller than

$$\epsilon + \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq \frac{T}{\lambda}} |\mathbf{R}_{\lambda,\epsilon,t}| \right] + \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq \frac{T}{\lambda}} \left| \frac{\lambda}{2\epsilon} \int_0^t (dr - (d\mathbf{Q}_r)^2) e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} \right| \right], \quad (3.13)$$

where ϵ bounds $\mathbb{E}^{(\lambda)} [\epsilon e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_0}]$.

To bound the remainder term $\mathbf{R}_{\lambda,\epsilon,t}$ in (3.13), we may write

$$\begin{aligned}
\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq \frac{T}{\lambda}} |\mathbf{R}_{\lambda,\epsilon,t}| \right] &\leq \frac{\lambda^{\frac{3}{2}}}{3\epsilon^2} \mathbb{E}^{(\lambda)} \left[\sum_{n=1}^{\mathcal{N}_{\frac{T}{\lambda}}} |\Delta \mathbf{Q}_{t_n}|^3 \right] = \frac{\lambda^{\frac{3}{2}}}{3\epsilon^2} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr \mathcal{K}_{\lambda,3}(X_r, P_r) \right] \\
&\leq C_1 \frac{\lambda^{\frac{3}{2}}}{3\epsilon^2} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr (1 + \lambda \mathbf{Q}_r)^4 \right] \leq C'_1 T \frac{\lambda^{\frac{1}{2}}}{\epsilon^2} = O(\lambda^{\frac{1}{2}}),
\end{aligned}$$

where the first inequality is by Part (1) of Proposition 3.3, and the $C'_1 > 0$ in the second inequality exists by bounding the moments of $Q_r = (2H_r)^{\frac{1}{2}}$, $0 \leq r \leq \frac{T}{\lambda}$ using Lemma 3.4.

By adding and subtracting $\int_0^t dr \mathcal{K}_{\lambda,2}(X_r, P_r)$ in the expression for the last term in (3.13) and using the triangle inequality,

$$\begin{aligned}
\mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq \frac{T}{\lambda}} \left| \frac{\lambda}{2\epsilon} \int_0^t (dr - (d\mathbf{Q}_r)^2) e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} \right| \right] &\leq \mathbb{E}^{(\lambda)} \left[\frac{\lambda}{2\epsilon} \int_0^{\frac{T}{\lambda}} dr |1 - \mathcal{K}_{\lambda,2}(X_r, P_r)| \right] \\
&\quad + \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq \frac{T}{\lambda}} \left| \frac{\lambda}{2\epsilon} \int_0^t (dr \mathcal{K}_{\lambda,2}(X_r, P_r) - (d\mathbf{Q}_r)^2) e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r^-}} \right| \right].
\end{aligned}$$

The first term on the right side is smaller than

$$\begin{aligned}
\mathbb{E}^{(\lambda)} \left[\frac{\lambda}{2\epsilon} \int_0^{\frac{T}{\lambda}} dr |1 - \mathcal{K}_{\lambda,2}(X_r, P_r)| \right] &\leq C_1 \frac{1}{\epsilon} \mathbb{P}^{(\lambda)} [\mathbf{T}_{\frac{T}{\lambda}}] + C_2 \frac{\lambda^{\frac{1}{2}}}{2\epsilon} + C_3 \frac{\lambda}{2\epsilon} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr \chi(\mathbf{Q}_r \geq \lambda^{-\frac{3}{4}}) (1 + \lambda \mathbf{Q}_r)^3 \right] \quad (3.14)
\end{aligned}$$

for some $C_1, C_2, C_3 > 0$, where $\mathbf{T}_t = \lambda \int_0^t dr \chi(\mathbf{Q}_r \leq \lambda^{-\frac{3}{8}})$, and the three terms on the right correspond to the parts of the trajectory such that $\mathbf{Q}_r \leq \lambda^{-\frac{3}{8}}$, $\lambda^{-\frac{3}{8}} \leq \mathbf{Q}_r \leq \lambda^{-\frac{3}{4}}$, and $\lambda^{-\frac{3}{4}} \leq$

\mathbf{Q}_r . For the first and second terms on the right side of (3.14), we have applied Part (1) of Proposition 3.3. The first term is $O(\lambda^{\frac{1}{8}})$ by Lemma 3.5. For the last term on the right side of (3.14), we can apply Cauchy-Schwarz and an analogous argument to that at the end of Part (I).

Moreover, the expression $\int_0^t (dr \mathcal{K}_{\lambda,2}(X_r, P_r) - (d\mathbf{Q}_r)^2) e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r-}}$ is a martingale with predictable quadratic variation

$$\int_0^t dr \left(\mathcal{K}_{\lambda,4}(X_r, P_r) - \frac{\mathcal{K}_{\lambda,2}^2(X_r, P_r)}{\mathcal{E}_\lambda(P_r)} \right) e^{-2\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_r}.$$

Hence, by Doob's inequality

$$\begin{aligned} & \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq \frac{T}{\lambda}} \left| \frac{\lambda}{2\epsilon} \int_0^t (dr \mathcal{K}_{\lambda,2}(X_r, P_r) - (d\mathbf{Q}_r)^2) e^{-\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_{r-}} \right|^2 \right]^{\frac{1}{2}} \\ & \leq \frac{\lambda}{2\epsilon} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr \left(\mathcal{K}_{\lambda,4}(X_r, P_r) - \frac{\mathcal{K}_{\lambda,2}^2(X_r, P_r)}{\mathcal{E}_\lambda(P_r)} \right) e^{-2\epsilon^{-1}\lambda^{\frac{1}{2}}\mathbf{Q}_r} \right]^{\frac{1}{2}} \\ & \leq \frac{\lambda}{2\epsilon} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr \mathcal{K}_{\lambda,4}(X_r, P_r) \right]^{\frac{1}{2}} \leq C_1 \frac{\lambda}{2\epsilon} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr (1 + \lambda \mathbf{Q}_r)^{n+1} \right]^{\frac{1}{2}} \leq C'_1 \frac{T\lambda^{\frac{1}{2}}}{\epsilon}. \end{aligned}$$

The third inequality holds for some C_1 by Part (1) of Proposition 3.3 (and $|p| \leq 2^{\frac{1}{2}} H^{\frac{1}{2}}(x, p)$), and the fourth inequality is for some C'_1 by Lemma 3.4.

4 The martingale problem

In the lemma below, we consider the class of process pairs $(\mathbf{p}, \mathbf{m}) \in \mathbb{R}^2$ such that the first component is an Ornstein-Uhlenbeck process and the second component is a continuous martingale. With the additional criterion that $\langle \mathbf{m} \rangle$ is the local time of the process \mathbf{p} at zero, Lemma 4.1 states that the law for the pair (\mathbf{p}, \mathbf{m}) is determined uniquely as $(\mathbf{p}, \mathbf{B}_l)$, where \mathbf{B} is a standard Brownian motion independent of \mathbf{p} . For the process inverse \mathbf{s} of \mathbf{l} , we can immediately observe that process $\mathbf{B}_t := \mathbf{m}_{\mathbf{s}_t}$ is a Brown motion, since it is a continuous martingale with quadratic variation t . Thus the question concerns the independence of \mathbf{B} from \mathbf{p} . Lemma 4.1 is a formulation of the *martingale problem* in the sense of [13]. For example, a standard Brownian motion is the unique continuous martingale \mathbf{m} satisfying that $\mathbf{m}_t^2 - t$ is a martingale. Our criterion could be formulated analogously by demanding that

$$\mathbf{m}_t^2 - \mathbf{l}_t$$

is a martingale. The proof of the Lemma makes use of the fact that \mathbf{l} almost surely makes all of its movement on a set of times having measure zero. If we only needed to show that (\mathbf{l}, \mathbf{m}) with the condition above necessarily has the law of $(\mathbf{l}, \mathbf{B}_l)$ for \mathbf{B} independent of \mathbf{l} , then we could apply the argument in [12, Thm. 4.21], since \mathbf{l} is the process inverse of the one-sided Levy process \mathbf{s} . However, \mathbf{p} contains information that \mathbf{l} does not, so there is the logical possibility that \mathbf{p} and \mathbf{B} are still dependent.

Lemma 4.1. *Consider a process $(\mathbf{p}, \mathbf{m}) \in \mathbb{R}^2$ and let \mathbb{F}_t be the filtration generated by it. Let \mathbf{p} be a copy of the Ornstein-Uhlenbeck process satisfying the Markov property with respect to \mathbb{F}_t*

and \mathbf{l} be the local time of \mathbf{p} at zero. Moreover, let \mathbf{m} be continuous, a martingale with respect to \mathbb{F}_t , and have predictable quadratic variation satisfying $\langle \mathbf{m} \rangle = \mathbf{l}$. It follows that (\mathbf{p}, \mathbf{m}) is equal in law to $(\mathbf{p}, \mathbf{B}_l)$, where \mathbf{B} is a standard Brownian motion independent of \mathbf{p} .

Proof. By definition, the process \mathbf{p} satisfies the Langevine equation $d\mathbf{p}_t = -\frac{1}{2}\mathbf{p}_t dt + d\mathbf{B}'_t$ for a standard Brownian motion \mathbf{B}' . Since \mathbf{p} satisfies the Markov property with respect \mathbb{F}_t , the Brownian motion \mathbf{B}' must also. We denote the right-continuous process inverse of \mathbf{l} by \mathbf{s} . The time-changed martingale $\mathbf{B}_t = \mathbf{m}_{\mathbf{s}_t}$ is continuous and has quadratic variation $\langle \mathbf{B} \rangle_t = t$, and is thus a copy of Brownian motion. We will construct a family of processes $\mathbf{p}^{(\epsilon)}$ such that

(I). $\mathbf{p}^{(\epsilon)}$ is independent of \mathbf{B} for each $\epsilon > 0$.

(II). As $\epsilon \rightarrow 0$, $\mathbb{E}[\sup_{0 \leq t \leq T} |\mathbf{p}_t^{(\epsilon)} - \mathbf{p}_t|] = O(\epsilon^{\frac{1}{2}-\delta})$ for any $\delta > 0$.

The above statements imply that the processes \mathbf{B} and \mathbf{p} are independent. Since \mathbf{l} is the process inverse of \mathbf{s} , $\mathbf{m}_t = \mathbf{B}_{\mathbf{s}_t}$. Thus (I) and (II) imply the result.

(I). First, we give definitions which are prerequisite to defining $\mathbf{p}^{(\epsilon)}$. If $|\mathbf{p}_0| < \epsilon$, let the stopping times $\varsigma_n, \varsigma'_n$ be defined such that $\varsigma_0 = \varsigma'_0 = \varsigma'_1 = 0$ and

$$\varsigma'_n = \min\{r \in (\varsigma_{n-1}, \infty) \mid |\mathbf{p}_r| \leq \frac{1}{2}\epsilon\}, \quad \varsigma_n = \min\{r \in (\varsigma'_n, \infty) \mid |\mathbf{p}_r| \geq \epsilon\},$$

and \mathbf{n}_t is the number of ς_n up to time t . If $|\mathbf{p}_0| \geq \epsilon$, then we use the same recursive definition with $\varsigma_0 = \varsigma'_0 = 0$. The intervals $[\varsigma'_n, \varsigma_n]$, $n \geq 0$ and $[\varsigma_n, \varsigma'_{n+1}]$, $n \geq 1$ will be referred to as the incursions and excursions respectively. Let τ_t be the hitting time that

$$t = \tau_t - \varsigma_{\mathbf{n}_{\tau_t}} + \sum_{n=0}^{\mathbf{n}_{\tau_t}-1} \varsigma'_{n+1} - \varsigma_n.$$

In other terms, τ_t is the first time that the total excursion time sums up to t .

Define another copy of Brownian motion $\mathbf{B}^{(\epsilon)}$

$$\mathbf{B}_t^{(\epsilon)} = \mathbf{B}'_{\tau_t} - \mathbf{B}'_{\varsigma_{\mathbf{n}_{\tau_t}}} + \sum_{n=0}^{\mathbf{n}_{\tau_t}-1} \mathbf{B}'_{\varsigma'_{n+1}} - \mathbf{B}'_{\varsigma_n}.$$

Define $\mathbf{p}^{(\epsilon)}$ and $\tilde{\mathbf{p}}^{(\epsilon)}$ to be the solutions of the Langevine equations

$$\begin{aligned} d\mathbf{p}_t^{(\epsilon)} &= -\frac{1}{2}\mathbf{p}_t^{(\epsilon)} dt + d\mathbf{B}_t^{(\epsilon)}, \\ d\tilde{\mathbf{p}}_t^{(\epsilon)} &= \chi(t \in \cup_{n=0}^{\infty} [\varsigma_n, \varsigma'_{n+1}]) \left(-\frac{1}{2}\tilde{\mathbf{p}}_t^{(\epsilon)} dt + d\mathbf{B}'_t \right), \end{aligned}$$

with $\mathbf{p}_0^{(\epsilon)} = \tilde{\mathbf{p}}_0^{(\epsilon)} = \mathbf{p}_0$. We will use the process $\tilde{\mathbf{p}}^{(\epsilon)}$ as an intermediary between $\mathbf{p}^{(\epsilon)}$ and \mathbf{p} in (II).

We claim that our construction makes the Brownian motion $\mathbf{B}^{(\epsilon)}$ independent of \mathbf{B} and thus $\mathbf{p}^{(\epsilon)}$ is also independent of \mathbf{B} . Construct the stopping time γ_t and the martingale $\mathbf{m}^{(\epsilon)}$ such that

$$t = \gamma_t - \varsigma_{\mathbf{n}_{\gamma_t}} + \sum_{n=1}^{\mathbf{n}_{\gamma_t}-1} \varsigma_n - \varsigma'_n \quad \text{and} \quad \mathbf{m}_t^{(\epsilon)} = \mathbf{m}_{\gamma_t} - \mathbf{m}_{\varsigma_{\mathbf{n}_{\gamma_t}}} + \sum_{n=1}^{\mathbf{n}_{\gamma_t}-1} \mathbf{m}_{\varsigma_n} - \mathbf{m}_{\varsigma'_n}.$$

Analogously to τ_t , the above means that γ_t is the first time that the duration of all the incursions sums up to t . The martingale $\mathbf{m}^{(\epsilon)}$ is a time-change of \mathbf{m} with $\mathbf{m}_{\gamma_t} = \mathbf{m}_t^{(\epsilon)}$ in which a portion of the pauses during which $\langle m \rangle = \mathbf{l}$ remains constant have been cut out. Since only pauses have been cut out, $\sigma(\mathbf{m}^{(\epsilon)})$ contains all of the information regarding \mathbf{B} . However, the σ -algebras $\sigma(\mathbf{B}^{(\epsilon)})$ and $\sigma(\mathbf{m}^{(\epsilon)})$ are independent. This follows since $\sigma(\mathbf{B}^{(\epsilon)})$ has no information about the incursions—including their durations, and vice versa for $\sigma(\mathbf{m}^{(\epsilon)})$.

(II). By the triangle inequality,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\mathbf{p}_t^{(\epsilon)} - \mathbf{p}_t|\right] \leq \mathbb{E}\left[\sup_{0 \leq t \leq T} |\mathbf{p}_t^{(\epsilon)} - \tilde{\mathbf{p}}_t^{(\epsilon)}| + \mathbb{E}\left[\sup_{0 \leq t \leq T} |\tilde{\mathbf{p}}_t^{(\epsilon)} - \mathbf{p}_t|\right]\right]. \quad (4.1)$$

We bound the first and second terms on the right side of (4.1) in (i) and (ii) below. First we show that $\mathbb{E}[\tau_T - T] = O(\epsilon)$, which is used in both parts. A Riemann over-sum using that $4n \geq 2(n+1)$ for $n \geq 1$ gives the first inequality below.

$$\begin{aligned} \mathbb{E}[\tau_T - T] &\leq \mathbb{E}[\tau_T \wedge (2T) - T] + 4T \sum_{n=1}^{\infty} \mathbb{P}[\tau_T \geq 2nT] \\ &\leq \mathbb{E}[\tau_T \wedge (2T) - T] + 4T \sum_{n=1}^{\infty} \left(\sup_{q \in \mathbb{R}} \mathbb{P}_q[\tau_T \geq 2T] \right)^n \\ &= \mathbb{E}[\tau_T \wedge (2T) - T] + 4T \frac{\mathbb{P}_0[\tau_T > 2T]}{1 - \mathbb{P}_0[\tau_T > 2T]} = O(\epsilon). \end{aligned} \quad (4.2)$$

In order for the event $\tau_T > 2nT$ to occur, the random walker must fail to accumulate a duration T of excursion time over n disjoint intervals of length $2T$. Thus $\mathbb{P}[\tau_T \geq 2nT] \leq (\sup_{q \in \mathbb{R}} \mathbb{P}_q[\tau_T \geq 2T])^n$, as we have used in the second inequality. The equality in (4.2) is from summing the geometric series, and since $\mathbb{P}_q[\tau_T \geq 2T]$ is minimized for $q = 0$. The starting point $q = 0$ maximizes the probability that τ_T is large (e.g. $\geq 2T$), since the process must travel the furthest to attain a value $|\mathbf{p}_t| \geq \epsilon$ in which the excursion clock may begin to run.

To show the order equality (4.2), we show that $\mathbb{P}_0[\tau_T > 2T]$ and $\mathbb{E}[\tau_T \wedge (2T) - T]$ are $O(\epsilon)$. We first note that

$$\begin{aligned} \mathbb{P}_0[\tau_T \geq 2T] &\leq \mathbb{P}_0\left[\int_0^{2T} dr \chi(|\mathbf{p}_r| \leq \epsilon) \geq T\right] \\ &\leq \frac{1}{T} \mathbb{E}_0\left[\int_0^{2T} dr \chi(|\mathbf{p}_r| \leq \epsilon)\right] = \frac{1}{T} \int_0^{2T} dt \int_{[-\epsilon, \epsilon]} dq \frac{e^{-\frac{q^2}{2\omega_t}}}{(2\pi\omega_t)^{\frac{1}{2}}} = O(\epsilon), \end{aligned}$$

where $\omega_t = 1 - e^{-\frac{1}{2}t}$. The first inequality uses that the event $\tau_T \geq 2T$ implies the event $\int_0^{2T} dr \chi(|\mathbf{p}_r| \leq \epsilon) \geq T$, since the incursions have $|\mathbf{p}_r| \leq \epsilon$. The second inequality is Jensen's, and the second equality uses that the density $\frac{e^{-\frac{q^2}{2\omega_t}}}{(2\pi\omega_t)^{\frac{1}{2}}}$ is the explicit solution to Ornstein-Uhlenbeck forward equation (i.e. Kramer's equation) starting from zero. The other term is similar

$$\begin{aligned} \mathbb{E}[\tau_T \wedge (2T) - T] &\leq \mathbb{E}\left[\int_0^{2T} dr \chi(|\mathbf{p}_r| \leq \epsilon)\right] \\ &\leq \mathbb{E}_0\left[\int_0^{2T} dr \chi(|\mathbf{p}_r| \leq \epsilon)\right] = \int_0^{2T} dt \int_{[-\epsilon, \epsilon]} dq \frac{e^{-\frac{q^2}{2\omega_t}}}{(2\pi\omega_t)^{\frac{1}{2}}} = O(\epsilon). \end{aligned}$$

(i). Notice that $\mathfrak{p}^{(\epsilon)}$ is a stochastic time-change of $\tilde{\mathfrak{p}}^{(\epsilon)}$ with $\mathfrak{p}_t^{(\epsilon)} = \tilde{\mathfrak{p}}_{\tau_t}^{(\epsilon)}$. Thus the first term on right side of (4.1) is smaller than

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\mathfrak{p}_t^{(\epsilon)} - \tilde{\mathfrak{p}}_t^{(\epsilon)}| \right] &\leq \mathbb{E} \left[\sup_{\substack{0 \leq r \leq \tau_T - T \\ 0 \leq t \leq T}} |\mathfrak{p}_{t+r}^{(\epsilon)} - \mathfrak{p}_t^{(\epsilon)}| \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sup_{\substack{0 \leq r \leq \tau_T - T \\ 0 \leq t \leq T}} |\mathfrak{p}_{t+r}^{(\epsilon)} - \mathfrak{p}_t^{(\epsilon)}| \mid \tau_T - T \right] \right] = \mathbb{E} \left[\delta_{\tau_T - T}(v) \mathbb{E} \left[\sup_{\substack{0 \leq r \leq v \\ 0 \leq t \leq T}} |\mathfrak{p}_{t+r}^{(\epsilon)} - \mathfrak{p}_t^{(\epsilon)}| \right] \right] \\ &\leq \mathbb{E} \left[(1 - e^{-\frac{1}{2}(\tau_T - T)}) \sup_{0 \leq t \leq \tau_T} |\mathfrak{p}_t^{(\epsilon)}| \right] + \mathbb{E} \left[\delta_{\tau_T - T}(v) \mathbb{E} \left[\sup_{\substack{0 \leq r \leq v \\ 0 \leq t \leq T}} \left| \int_t^{t+r} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(t+r-s)} \right| \right] \right] \quad (4.3) \end{aligned}$$

The second equality follows, since the process $\mathfrak{p}^{(\epsilon)}$ and the difference $\tau_T - T$ are independent. For the last inequality, we have used the triangle inequality with the explicit form in the first equality below:

$$\begin{aligned} \mathfrak{p}_{t+r}^{(\epsilon)} - \mathfrak{p}_t^{(\epsilon)} &= (e^{-\frac{1}{2}r} - 1)\mathfrak{p}_t^{(\epsilon)} + \int_t^{t+r} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(r+t-s)} \\ &= (e^{-\frac{1}{2}r} - 1)\mathfrak{p}_t^{(\epsilon)} + \mathbf{B}_{t+r}^{(\epsilon)} - \mathbf{B}_t^{(\epsilon)} - \frac{1}{2} \int_t^{t+r} ds (\mathbf{B}_{s+t}^{(\epsilon)} - \mathbf{B}_t^{(\epsilon)}) e^{-\frac{1}{2}(r+t-s)}. \quad (4.4) \end{aligned}$$

The second equality is Ito's product rule. Note that for $m \geq 1$

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq v \leq r} \left| \int_t^{t+v} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(t+r-s)} \right|^{2m} \right] &\leq 2^m \mathbb{E} \left[\sup_{0 \leq v \leq r} \left| \mathbf{B}_{t+v}^{(\epsilon)} - \mathbf{B}_t^{(\epsilon)} \right|^{2m} \right] \\ &\leq \left(\frac{4m}{2m-1} \right)^{2m} \mathbb{E} \left[\left| \mathbf{B}_{t+r}^{(\epsilon)} - \mathbf{B}_t^{(\epsilon)} \right|^{2m} \right] = m! \left(\frac{4m}{2m-1} \right)^{2m} r^m. \quad (4.5) \end{aligned}$$

The first inequality comes from rewriting $\int_t^{t+v} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(t+r-s)}$ as in (4.4), applying the triangle inequality, and using that $\int_t^{t+r} ds e^{-\frac{1}{2}(t+r-s)} \leq 2$. The second inequality is Doob's, and the last is a computation of the Gaussian moment.

For the first term on the right side of (4.3), we have following routine inequalities

$$\begin{aligned} \mathbb{E} \left[(1 - e^{-\frac{1}{2}(\tau_T - T)}) \sup_{0 \leq t \leq \tau_T} |\mathfrak{p}_t^{(\epsilon)}| \right] &\leq \mathbb{E} \left[(1 - e^{-\frac{1}{2}(\tau_T - T)})^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq \tau_T} |\mathfrak{p}_t^{(\epsilon)}|^2 \right]^{\frac{1}{2}} \\ &\leq \mathbb{E} \left[(\tau_T - T) \wedge 1 \right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq \tau_T} |\mathfrak{p}_t^{(\epsilon)}|^2 \right]^{\frac{1}{2}} \\ &\leq C\epsilon^{\frac{1}{2}} \mathbb{E} [|\mathfrak{p}_0|^2]^{\frac{1}{2}} + C\epsilon^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq \tau_T} \left| \int_0^t d\mathbf{B}_r^{(\epsilon)} e^{-\frac{1}{2}(t-r)} \right|^2 \right]^{\frac{1}{2}} \\ &\leq C\epsilon^{\frac{1}{2}} \mathbb{E} [|\mathfrak{p}_0|^2]^{\frac{1}{2}} + C\epsilon^{\frac{1}{2}} 2\mathbb{E} [\tau_T]^{\frac{1}{2}} = O(\epsilon). \end{aligned}$$

The last inequality follows from the independence of τ_T and the Brownian motion $\mathbf{B}^{(\epsilon)}$ and (4.5).

Now we bound the second term on the right side of (4.3). We have the following relations

$$\begin{aligned}
\mathbb{E} \left[\sup_{\substack{0 \leq r \leq v \\ 0 \leq t \leq T}} \left| \int_0^r d\mathbf{B}_{t+s}^{(\epsilon)} e^{-\frac{1}{2}(r-s)} \right| \right] &= \mathbb{E} \left[\sup_{\substack{0 \leq r \leq v \\ 0 \leq z+r \leq T+v}} \left| \int_z^{z+r} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(z+r-s)} \right| \right] \\
&\leq 2\mathbb{E} \left[\sup_{0 \leq n \leq \lfloor \frac{T+v}{v} \rfloor} \sup_{0 \leq r \leq v} \left| \int_{nv}^{nv+r} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(nv+r-s)} \right| \right] \\
&\leq 2\mathbb{E} \left[\sum_{n=0}^{\lfloor \frac{T+v}{v} \rfloor} \sup_{0 \leq r \leq v} \left| \int_z^{z+r} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(z+r-s)} \right|^{2m} \right]^{\frac{1}{2m}} \\
&= 2 \left[\frac{T+v}{v} \right]^{\frac{1}{2m}} \mathbb{E} \left[\sup_{0 \leq r \leq v} \left| \int_0^r d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(r-s)} \right|^{2m} \right]^{\frac{1}{2m}} \\
&\leq 2(m!)^{\frac{1}{2m}} \frac{4m}{2m-1} \left[\frac{T+v}{v} \right]^{\frac{1}{2m}} v^{\frac{1}{2}} < 6m^{\frac{1}{2}} |T+v|^{\frac{1}{2m}} |v|^{\frac{m-1}{2m}},
\end{aligned}$$

where the last inequality is for $m \geq 1$ large enough. The second inequality is $(\sup_n a_n)^{2m} \leq \sum_n a_n^{2m}$ followed by Jensen's inequality, the second equality is from the stationarity of the increments for $\mathbf{B}^{(\epsilon)}$, and the third inequality is from (4.5). With the above

$$\begin{aligned}
\mathbb{E} \left[\delta_{\tau_T - T}(v) \mathbb{E} \left[\sup_{\substack{0 \leq r \leq v \\ 0 \leq t \leq T}} \left| \int_t^{t+r} d\mathbf{B}_s^{(\epsilon)} e^{-\frac{1}{2}(t+r-s)} \right| \right] \right] &\leq 6m^{\frac{1}{2}} \mathbb{E} \left[|\tau_T|^{\frac{1}{2m}} |\tau_T - T|^{\frac{m-1}{2m}} \right] \\
&\leq 6m^{\frac{1}{2}} \mathbb{E} \left[\tau_T^{\frac{1}{m+1}} \right]^{\frac{m+1}{2m}} \mathbb{E} \left[\tau_T - T \right]^{\frac{m-1}{2m}} = O(\epsilon^{\frac{m-1}{2m}}),
\end{aligned}$$

where the second inequality is Holder's. The value m can be picked to make the power of ϵ arbitrarily close to $\frac{1}{2}$.

(ii). Notice that \mathfrak{p} and $\tilde{\mathfrak{p}}^{(\epsilon)}$ satisfy the equations

$$\mathfrak{p}_t = e^{-\frac{1}{2}t} \mathfrak{p}_0 + \int_0^t d\mathbf{B}'_r e^{-\frac{1}{2}(t-r)} \tag{4.6}$$

$$\tilde{\mathfrak{p}}_t^{(\epsilon)} = e^{-\frac{1}{2}t} \mathfrak{p}_0 + \int_0^t d\mathbf{B}'_r \chi_r^{(\epsilon)} e^{-\frac{1}{2}(t-r)} + \int_0^t dr \tilde{\mathfrak{p}}_r^{(\epsilon)} e^{-\frac{1}{2}(t-r)} (1 - \chi_r^{(\epsilon)}), \tag{4.7}$$

where $\chi_r^{(\epsilon)} = \chi(r \in \cup_{n=0}^{\infty} [\varsigma_n, \varsigma'_{n+1}])$. The Ito product rule for the martingale $\int_0^t d\mathbf{B}'_r (1 - \chi_r^{(\epsilon)})$ gives

$$\int_0^t d\mathbf{B}'_r (1 - \chi_r^{(\epsilon)}) e^{-\frac{1}{2}(t-r)} = \int_0^t d\mathbf{B}'_r (1 - \chi_r^{(\epsilon)}) - \frac{1}{2} \int_0^t dr e^{-\frac{1}{2}(t-r)} \int_0^r d\mathbf{B}'_s (1 - \chi_s^{(\epsilon)}). \tag{4.8}$$

Similarly to (4.5),

$$\begin{aligned}
\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t d\mathbf{B}'_r (1 - \chi_r^{(\epsilon)}) e^{-\frac{1}{2}(t-r)} \right|^2 \right] &\leq 4\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t d\mathbf{B}'_t (1 - \chi_t^{(\epsilon)}) \right|^2 \right] \\
&\leq 16\mathbb{E} \left[\left| \int_0^T d\mathbf{B}'_t (1 - \chi_t^{(\epsilon)}) \right|^2 \right] = 16\mathbb{E} \left[\int_0^T dt (1 - \chi_t^{(\epsilon)}) \right] \leq 16\mathbb{E} \left[\int_0^T dt \chi(|\mathfrak{p}_t| < \epsilon) \right] \\
&\leq 16\mathbb{E}_0 \left[\int_0^T dt \chi(|\mathfrak{p}_t| < \epsilon) \right] = \int_0^T dt \int_{[-\epsilon, \epsilon]} dq \frac{e^{-\frac{q^2}{2\omega_t}}}{(2\pi\omega_t)^{\frac{1}{2}}} = O(\epsilon).
\end{aligned} \tag{4.9}$$

The first inequality is from (4.8) with the triangle inequality, and the second inequality is Doob's. The fourth inequality uses that the initial value $\mathbf{p}_0 = 0$ will maximize the expectation of the quantity $\int_0^T dt \chi_t(|\mathbf{p}_t| < \epsilon)$.

Using (4.6) and (4.7) with the triangle inequality, we have the first inequality below:

$$\begin{aligned}
& \mathbb{E} \left[\sup_{0 \leq t \leq T} |\tilde{\mathbf{p}}_t^{(\epsilon)} - \mathbf{p}_t| \right] \\
& \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t d\mathbf{B}'_r e^{-\frac{1}{2}(t-r)} (1 - \chi_r^{(\epsilon)}) \right| \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t dr \tilde{\mathbf{p}}_r^{(\epsilon)} e^{-\frac{1}{2}(t-r)} (1 - \chi_r^{(\epsilon)}) \right| \right] \\
& \leq O(\epsilon) + \mathbb{E} \left[\sup_{0 \leq t \leq T} |\tilde{\mathbf{p}}_t^{(\epsilon)}|^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\left(\int_0^T dt (1 - \chi_t^{(\epsilon)}) \right)^2 \right]^{\frac{1}{2}} \\
& \leq O(\epsilon) + T^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\mathbf{p}_r|^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T dt (1 - \chi_t^{(\epsilon)}) \right]^{\frac{1}{2}} = O(\epsilon^{\frac{1}{2}}). \tag{4.10}
\end{aligned}$$

The second inequality uses (4.9) for the first term and Holder's equality twice for the second term. The second inequality follows from the fact that $\tilde{\mathbf{p}}_{\tau_t}^{(\epsilon)}$ has the same law as \mathbf{p}_t and $\tau_t \geq t$. In other words, \mathbf{p} has the same law as a sped-up version of $\tilde{\mathbf{p}}^{(\epsilon)}$. Finally, $\mathbb{E} \left[\int_0^T dt (1 - \chi_t^{(\epsilon)}) \right] = O(\epsilon)$ by (4.9).

□

5 Proof of Theorem 1.1

Let us define (or recall) the following notations:

- $\tilde{S}_t = (S_t, Z_t)$ State of the split process at time t .
- $\tau_m \in \mathbb{R}^+$ m th partition time.
- $\mathbf{N}_t \in \mathbb{N}$ Number of non-zero partition times up to time t .
- $R_m \in \mathbb{R}^+$ Beginning time of the m th life cycle.
- $\tilde{N}_t \in \mathbb{N}$ Number of returns to the atom up to time t .
- \mathcal{F}_t Information up to time t for the original process S_r and the τ_m .
- $\tilde{\mathcal{F}}_t$ Information up to time t for the split process \tilde{S}_t and the τ_m .
- $\tilde{\mathcal{F}}'_t$ Information for \tilde{S}_t and the τ_m before time R_{n+1} , where $R'_n \leq t < R'_{n+1}$,

Let the constant $\mathbf{u} > 0$, the function $h : \Sigma \rightarrow [0, 1]$, and measure $\tilde{\nu}$ on $\tilde{\Sigma}$ be defined as in Section 2. Define $v_\lambda > 0$ as

$$\begin{aligned}
v_\lambda & := 2\tilde{\mathbb{E}}_{\tilde{\nu}}^{(\lambda)} \left[\int_0^{R_1} dr \frac{dV}{dx}(X_r) \int_r^{R_2} dr' \frac{dV}{dx}(X_{r'}) \right] \\
& = \frac{2 \int_{\Sigma} dx dp e^{-\lambda H(x,p)} \frac{dV}{dx}(x) (\mathfrak{R}^{(\lambda)} \frac{dV}{dx})(x, p)}{\int_{\Sigma} dx dp e^{-\lambda H(x,p)} h(x, p)},
\end{aligned}$$

where the equality holds by [6, Prop. 4.4]. Notice that v_λ is formally equal to $\frac{\kappa}{\mathbf{u}}$ for $\lambda = 0$, since the numerator is the formal Green-Kubo expression (1.8) and the denominator is $\mathbf{u} = \int_S ds h(s)$.

The value $v_0 > 0$ is a well-defined by [6, Prop. 4.14], and we can give a rigorous definition for κ as

$$\kappa := \mathbf{u} v_0.$$

The following proposition is from [6, Prop. 4.16] and [6, Lem. 4.17]. The martingale \tilde{M}_t was defined in Lemma 2.1.

Proposition 5.1.

1. For the split statistics, $\tilde{N}_t - \sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n})$ is a martingale with respect to the filtration $\tilde{\mathcal{F}}_t$.
For the original statistics, $\sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n}) - \int_0^t dr h(S_r)$ is a martingale with respect to \mathcal{F}_t .
In particular,

$$\tilde{\mathbb{E}}^{(\lambda)}[\tilde{N}_t] = \mathbb{E}^{(\lambda)}\left[\int_0^t dr h(S_r)\right].$$

2. As $\lambda \rightarrow 0$,

$$\tilde{\mathbb{E}}^{(\lambda)}\left[\sup_{0 \leq t \leq T} \left|\lambda^{\frac{1}{2}} \langle \tilde{M} \rangle_{\frac{t}{\lambda}} - \lambda^{\frac{1}{2}} v_\lambda \tilde{N}_{\frac{t}{\lambda}}\right|\right] = O(\lambda^{\frac{1}{4}}).$$

Also, for any $t \geq 0$, the expectations are equal $\tilde{\mathbb{E}}^{(\lambda)}[\langle \tilde{M} \rangle_t] = v_\lambda \tilde{\mathbb{E}}^{(\lambda)}[\tilde{N}_t]$.

The equality in Proposition 5.2 is from [6, Prop. 4.3] and is of a standard type for splitting constructions [21]. It states that the probability of the process being at the atom at time r , conditioned on r being a partition time (i.e. $\mathbf{N}_r = \mathbf{N}_{r^-} + 1$) and the entire past $\tilde{\mathcal{F}}_{r^-}$, is given by the value $h(S_r)$. Note that the value S_r is a.s. contained in $\tilde{\mathcal{F}}_{r^-}$, since a collision will a.s. not occur at the partition time r and thus $\lim_{v \nearrow r} S_v = S_r$.

Proposition 5.2.

$$\tilde{\mathbb{P}}^{(\lambda)}[Z_r = 1 \mid \tilde{\mathcal{F}}_{r^-}, \mathbf{N}_r - \mathbf{N}_{r^-} = 1] = h(S_r)$$

Our proof of Theorem 1.1 takes some inspiration from the proof of [12, Thm. 4.12] and relies heavily on [13].

[Proof of Theorem 1.1]

For the study of the pair $(\lambda^{\frac{1}{2}} P_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{4}} D_{\frac{\cdot}{\lambda}})$, we will begin by embedding the processes in the split statistics defined in Section 2. Let the martingale \tilde{M} be defined as in Lemma 2.1. In this proof, all convergences in law refer to the Skorokhod metric. The following points hold regarding the processes $\lambda^{\frac{1}{4}} D_{\frac{\cdot}{\lambda}}$ and $\lambda^{\frac{1}{4}} \tilde{M}_{\frac{\cdot}{\lambda}}$:

- (I). As $\lambda \rightarrow 0$,

$$\tilde{\mathbb{E}}^{(\lambda)}\left[\sup_{0 \leq t \leq T} \left|\lambda^{\frac{1}{4}} D_{\frac{t}{\lambda}} - \lambda^{\frac{1}{4}} \tilde{M}_{\frac{t}{\lambda}}\right|\right] \rightarrow 0.$$

- (II). As $\lambda \rightarrow 0$, the bracket process $\langle \tilde{M} \rangle_t$ satisfies

$$\tilde{\mathbb{E}}^{(\lambda)}\left[\sup_{0 \leq t \leq T} \left|\lambda^{\frac{1}{2}} \langle \tilde{M} \rangle_{\frac{t}{\lambda}} - \kappa \lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}}\right|\right] \rightarrow 0,$$

where $L_t = \mathbf{u}^{-1} \int_0^t dr h(X_r, P_r)$.

(III). The martingale $\lambda^{\frac{1}{4}}\tilde{M}_{\frac{t}{\lambda}}$ satisfies the Lindberg condition

$$\sup_{0 < \lambda \leq 1} \tilde{\mathbb{P}}^{(\lambda)} \left[\sup_{1 \leq r \leq \tilde{N}_{\frac{T}{\lambda}}} \left| \tilde{M}_r - \tilde{M}_{r^-} \right|^2 > \frac{\epsilon}{\lambda} \right] \longrightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Statements (I) and (III) have already been shown in the proof of [6, Thm. 4.18].

We will temporarily assume statement (II) and proceed with the main part of the proof. By (I), we may work with the pair $(\lambda^{\frac{1}{2}}P_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda}})$ rather than $(\lambda^{\frac{1}{2}}P_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{4}}D_{\frac{\cdot}{\lambda}})$. By Theorem 3.1 and (II), there is convergence in law

$$(\lambda^{\frac{1}{2}}P_{\frac{t}{\lambda}}, \lambda^{\frac{1}{2}}\langle \tilde{M} \rangle_{\frac{t}{\lambda}}) \xrightarrow{\mathcal{L}} (\mathbf{p}_t, \kappa \mathbf{l}_t), \quad (5.1)$$

as $\lambda \rightarrow 0$. It follows that the components $\lambda^{\frac{1}{2}}P_{\frac{\cdot}{\lambda}}$ and $\lambda^{\frac{1}{2}}\langle \tilde{M} \rangle_{\frac{\cdot}{\lambda}}$ are C -tight for $\lambda < 1$. By [13, Thm. VI.4.13], the family of martingales $\lambda^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda}}$ must be tight for $\lambda < 1$. The Lindberg condition (III) and [13, Prop. VI.3.26] guarantee that the family of martingales must be C -tight.

The triple $T^{(\lambda)} = (\lambda^{\frac{1}{2}}P_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{2}}\langle \tilde{M} \rangle_{\frac{\cdot}{\lambda}}, \lambda^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda}})$ is C -tight for $\lambda < 1$ by [13, Cor. VI.3.33], since all of the components are \tilde{C} -tight. By tightness, we may consider a subsequence $\lambda_n \rightarrow 0$ such that $T^{(\lambda_n)}$ converges in law to a limit $(\mathbf{p}, \mathbf{v}, \mathbf{m})$. The first two components \mathbf{p}, \mathbf{v} are the Ornstein-Uhlenbeck process and κ multiplied its the local time (i.e. $\mathbf{v} = \kappa \mathbf{l}$), respectively, by (5.1). We will argue that the third component \mathbf{m}_t must be a continuous martingale with respect to the filtration $\sigma(\mathbf{p}_r, \mathbf{m}_r; 0 \leq r \leq t)$ such that $\langle \mathbf{m} \rangle = \kappa \mathbf{l}$. The continuity of \mathbf{m} follows by the C -tightness of $\lambda^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda}}$. The process \mathbf{m} is a martingale with respect to $\sigma(\mathbf{p}_r, \mathbf{m}_r; 0 \leq r \leq t)$ by [13, Prop. IX.1.17], since $(\lambda_n^{\frac{1}{2}}P_{\frac{\cdot}{\lambda_n}}, \lambda_n^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda_n}})$ is adapted to the filtration $\tilde{\mathcal{F}}_t^{(\lambda_n)} := \tilde{\mathcal{F}}_{\frac{t}{\lambda_n}}$, the process $\lambda_n^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda_n}}$ is a martingale with respect to $\tilde{\mathcal{F}}_t^{(\lambda_n)}$ by Lemma 2.1, and the family of random variables $\lambda_n^{\frac{1}{4}}\tilde{M}_{\frac{t}{\lambda_n}}$ for $\lambda < 1$ and $t \in [0, T]$ is uniformly square integrable. To see the uniform square integrability, notice

$$\sup_{0 \leq t \leq T} \tilde{\mathbb{E}}^{(\lambda)} [(\lambda^{\frac{1}{4}}\tilde{M}_{\frac{t}{\lambda}})^2] = \tilde{\mathbb{E}}^{(\lambda)} [\lambda^{\frac{1}{2}}\langle \tilde{M} \rangle_{\frac{T}{\lambda}}] = v_\lambda \tilde{\mathbb{E}}^{(\lambda)} [\lambda^{\frac{1}{2}}\tilde{N}_{\frac{T}{\lambda}}] = v_\lambda \mathbb{E}^{(\lambda)} \left[\lambda^{\frac{1}{2}} \int_0^{\frac{T}{\lambda}} dr h(S_r) \right]. \quad (5.2)$$

The second and third equalities are by Part (2) and Part (1) of Proposition 5.1, respectively. The right side of (5.2) is uniformly bounded for $\lambda < 1$ by Theorem 3.1, and thus $\sup_{t \in [0, T]} \sup_{\lambda < 1} \tilde{\mathbb{E}}^{(\lambda)} [(\lambda^{\frac{1}{4}}\tilde{M}_{\frac{t}{\lambda}})^2]$ is finite. By [13, Cor. VI.6.7], the convergence $\lambda_n^{\frac{1}{4}}\tilde{M}_{\frac{\cdot}{\lambda_n}} \xrightarrow{\mathcal{L}} \mathbf{m}$ with the Lindberg condition (III) implies the joint convergence of the pair

$$(\lambda_n^{\frac{1}{2}}\langle \tilde{M} \rangle_{\frac{t}{\lambda_n}}, \lambda_n^{\frac{1}{4}}\tilde{M}_{\frac{t}{\lambda_n}}) \xrightarrow{\mathcal{L}} (\langle \mathbf{m} \rangle_t, \mathbf{m}_t).$$

For the above, we have used that the difference between $\lambda_n^{\frac{1}{2}}[\tilde{M}]_{\frac{t}{\lambda_n}}$ and $\lambda_n^{\frac{1}{2}}\langle \tilde{M} \rangle_{\frac{t}{\lambda_n}}$ is $O(\lambda_n^{\frac{1}{4}})$. Thus $\langle \mathbf{m} \rangle = \kappa \mathbf{l}$.

We have now learned what we could from the martingale \tilde{M} . By (I), we have shown that $(\lambda_n^{\frac{1}{2}}P_{\frac{\cdot}{\lambda_n}}, \lambda_n^{\frac{1}{4}}D_{\frac{\cdot}{\lambda_n}})$ (interpreted as the original processes) converges in law to a pair (\mathbf{p}, \mathbf{m}) as $n \rightarrow \infty$, where \mathbf{m} is a continuous martingale with respect to the filtration $\sigma(\mathbf{p}_r, \mathbf{m}_r; 0 \leq r \leq t)$ and $\langle \mathbf{m} \rangle = \kappa \mathbf{l}$. If we establish that \mathbf{p} satisfies the Markov property with respect to the filtration

$\sigma(\mathbf{p}_r, \mathbf{m}_r; 0 \leq r \leq t)$, then Lemma 4.1 states that the pair (\mathbf{p}, \mathbf{m}) must have the law of the process $(\mathbf{p}, \sqrt{\kappa} \mathbf{B}_l)$ for a copy of Brownian motion \mathbf{B} independent of \mathbf{p} . Since the pair $(\lambda^{\frac{1}{2}} P_{\bar{\lambda}}, \lambda^{\frac{1}{4}} D_{\bar{\lambda}})$ is tight for $\lambda < 1$, if the law $(\mathbf{p}, \sqrt{\kappa} \mathbf{B}_l)$ is the unique possible subsequential limit, this would establish the convergence in law of $(\lambda^{\frac{1}{2}} P_{\bar{\lambda}}, \lambda^{\frac{1}{4}} D_{\bar{\lambda}})$ as $\lambda \rightarrow 0$ to the process $(\mathbf{p}, \mathbf{B}_l)$.

To show that \mathbf{p} satisfies the Markov property with respect to the filtration $\sigma(\mathbf{p}_r, \mathbf{m}_r; 0 \leq r \leq t)$, it is enough to show that the trajectory \mathbf{p}_s , $s > t$ is independent of $\sigma(\mathbf{m}_r; 0 \leq r \leq t)$ when given $\sigma(\mathbf{p}_r; 0 \leq r \leq t)$, since the process \mathbf{p} satisfies the Markov property with respect to its own filtration. The triple $(\lambda^{\frac{1}{2}} X_{\frac{1}{\lambda n}}, \lambda^{\frac{1}{2}} P_{\frac{1}{\lambda n}}, \lambda^{\frac{1}{4}} D_{\frac{1}{\lambda n}})$ converges to $(0, \mathbf{p}, \mathbf{m})$, since the variable $X \in \mathbb{T} = [0, 1]$ is bounded. Moreover, $\sigma(\lambda^{\frac{1}{2}} X_{\frac{1}{\lambda n}}, \lambda^{\frac{1}{2}} P_{\frac{1}{\lambda n}}; 0 \leq r \leq t)$ contains the information in $\sigma(\lambda^{\frac{1}{4}} D_{\frac{1}{\lambda n}}; 0 \leq r \leq t)$, since D_t is defined by as a function of the Markov process (X_r, P_r) for $0 \leq r \leq t$. Thus the path $\lambda^{\frac{1}{2}} P_{\frac{s}{\lambda n}}$, $s > t$ is independent of $\sigma(\lambda^{\frac{1}{4}} D_{\frac{1}{\lambda n}}; 0 \leq r \leq t)$ when given $\sigma(\lambda^{\frac{1}{2}} X_{\frac{1}{\lambda n}}, \lambda^{\frac{1}{2}} P_{\frac{1}{\lambda n}}; 0 \leq r \leq t)$. This independence carries over into the limit $n \rightarrow \infty$, and thus \mathbf{p}_s for $s > t$ is independent of $\sigma(\mathbf{m}_r; 0 \leq r \leq t)$ when given the information $\sigma(\mathbf{p}_r; 0 \leq r \leq t)$.

The remainder of the proof is concerned with showing (II).

(II) By the triangle inequality,

$$\begin{aligned} \tilde{\mathbb{E}}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \langle \tilde{M} \rangle_{\frac{t}{\lambda}} - \kappa \lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}} \right| \right] &\leq \tilde{\mathbb{E}}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \langle \tilde{M} \rangle_{\frac{t}{\lambda}} - v_\lambda \lambda^{\frac{1}{2}} \tilde{N}_{\frac{t}{\lambda}} \right| \right] + |v_\lambda - \frac{\kappa}{\mathbf{u}}| \tilde{\mathbb{E}}^{(\lambda)} \left[\lambda^{\frac{1}{2}} \tilde{N}_{\frac{T}{\lambda}} \right] \\ &\quad \frac{\kappa}{\mathbf{u}} \tilde{\mathbb{E}}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \tilde{N}_{\frac{t}{\lambda}} - \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_{\frac{t}{\lambda}}} h(S_{\tau_n}) \right| \right] + \kappa \tilde{\mathbb{E}}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \mathbf{u}^{-1} \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_{\frac{t}{\lambda}}} h(S_{\tau_n}) - \lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}} \right| \right], \end{aligned} \quad (5.3)$$

where \mathbf{N}_t is the number of partition times up to time t . The first term on the right is $O(\lambda^{\frac{1}{4}})$ by Part (2) of Lemma 5.1. The second term is bounded through

$$|v_\lambda - \frac{\kappa}{\mathbf{u}}| \tilde{\mathbb{E}}^{(\lambda)} \left[\lambda^{\frac{1}{2}} \tilde{N}_{\frac{T}{\lambda}} \right] = |v_\lambda - \frac{\kappa}{\mathbf{u}}| \mathbb{E}^{(\lambda)} \left[\lambda^{\frac{1}{2}} \int_0^{\frac{T}{\lambda}} dr h(S_r) \right] = O(\lambda^{\frac{1}{40}}),$$

where we have used Part (1) of Proposition 5.1 for the equality. For the inequality (i.e. order equality), we have used Theorem 3.1 to get a uniform constant bound for the expectation over $\lambda < 1$, and Part (3) of [6, Prop. 4.14] which gives that $|v_\lambda - \frac{\kappa}{\mathbf{u}}| = O(\lambda^{\frac{1}{40}})$.

For the third term in (5.3),

$$\begin{aligned} \tilde{\mathbb{E}}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \lambda^{\frac{1}{2}} \tilde{N}_{\frac{t}{\lambda}} - \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_{\frac{t}{\lambda}}} h(S_{\tau_n}) \right| \right] &\leq 2 \tilde{\mathbb{E}}^{(\lambda)} \left[\left| \lambda^{\frac{1}{2}} \tilde{N}_{\frac{T}{\lambda}} - \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_{\frac{T}{\lambda}}} h(S_{\tau_n}) \right|^2 \right]^{\frac{1}{2}} \\ &= 2 \lambda^{\frac{1}{2}} \tilde{\mathbb{E}}^{(\lambda)} \left[\sum_{n=1}^{\mathbf{N}_{\frac{T}{\lambda}}} h(S_{\tau_n}) - h^2(S_{\tau_n}) \right]^{\frac{1}{2}} \leq 2 \lambda^{\frac{1}{2}} \tilde{\mathbb{E}}^{(\lambda)} \left[\sum_{n=1}^{\mathbf{N}_{\frac{T}{\lambda}}} h(S_{\tau_n}) \right]^{\frac{1}{2}} = 2 \lambda^{\frac{1}{2}} \tilde{\mathbb{E}}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr h(S_r) \right]^{\frac{1}{2}}, \end{aligned} \quad (5.4)$$

The first inequality uses Jensen's inequality and Doob's inequality, since

$$\tilde{N}_t - \sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n}) = \sum_{n=1}^{\mathbf{N}_t} \chi(Z_{\tau_n} = 1) - h(S_{\tau_n})$$

is a martingale with respect $\tilde{\mathcal{F}}_t$ by Proposition 5.1. The first equality in (5.4) follows because the quadratic variation of the martingale is $\sum_{n=1}^t (\chi(Z_{\tau_n} = 1) - h(S_{\tau_n}))^2$, and

$$\tilde{\mathbb{E}}[(\chi(Z_r = 1) - h(S_r))^2 \mid \tilde{\mathcal{F}}_{r^-}, \mathbf{N}_r - \mathbf{N}_{r^-} = 1] = h(S_r) - h^2(S_r),$$

by Proposition 5.2. For the second inequality, we discard $h^2(S_{\tau_n})$, and go from the split to the original statistics, since the argument of the expectation is well-defined there. Finally, the last equality holds, since the partition times τ_n occur with Poisson rate 1 independently of the process S_t .

The fourth term in (5.3) similar to the third. The process $\mathbf{u}^{-1} \sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n}) - L_t$ is well-defined in the original statistics and is a martingale with respect to the filtration \mathcal{F}_t by Proposition 5.1. With routine arguments

$$\begin{aligned} \tilde{\mathbb{E}}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \mathbf{u}^{-1} \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n}) - \lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}} \right| \right] &= \mathbb{E}^{(\lambda)} \left[\sup_{0 \leq t \leq T} \left| \mathbf{u}^{-1} \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n}) - \lambda^{\frac{1}{2}} L_{\frac{t}{\lambda}} \right| \right] \\ &\leq 2 \mathbb{E}^{(\lambda)} \left[\left| \mathbf{u}^{-1} \lambda^{\frac{1}{2}} \sum_{n=1}^{\mathbf{N}_T} h(S_{\tau_n}) - \lambda^{\frac{1}{2}} L_{\frac{T}{\lambda}} \right|^2 \right]^{\frac{1}{2}} = \lambda^{\frac{1}{2}} \mathbf{u}^{-1} \mathbb{E}^{(\lambda)} \left[\int_0^{\frac{T}{\lambda}} dr h^2(S_r) \right]^{\frac{1}{2}} = O(\lambda^{\frac{1}{4}}). \end{aligned}$$

The first inequality uses Jensen's and Doob's inequalities. The second equality uses that the predictable quadratic variation of $\mathbf{u}^{-1} \sum_{n=1}^{\mathbf{N}_t} h(S_{\tau_n}) - L_t$ is $\mathbf{u}^{-2} \int_0^t dr h^2(S_r)$, since the terms $h(S_{\tau_n})$ occur with Poisson rate 1 independently of the process S_t .

Acknowledgments

This work is supported by the European Research Council grant No. 227772.

A The limiting diffusion process

A.1 Local time at the origin for an Ornstein-Uhlenbeck process

Let \mathfrak{p} be the Ornstein-Uhlenbeck process satisfying the Langevine equation (1.3) and \mathfrak{l} be the corresponding local time at zero. For a discussion of local time for continuous semimartingales we refer to [14, Sec. 3.7], and for a list of many formulae related to the local time of an Ornstein-Uhlenbeck process we refer to [2]. As mentioned before, the local time is formally $\mathfrak{l}_t = \int_0^t dr \delta_0(\mathfrak{p}_r)$, and through a formal application of the Ito formula, it satisfies

$$\mathfrak{l}_t = |\mathfrak{p}_t| - |\mathfrak{p}_0| - \int_0^t dr \operatorname{sgn}(\mathfrak{p}_r) dB_r + \frac{1}{2} \int_0^t dr |\mathfrak{p}_r| dr,$$

where $\operatorname{sgn} : \mathbb{R} \rightarrow \{\pm 1\}$ is the sign function. The above is one of the Tanaka-Meyer formulas. The process \mathfrak{l} is a continuous increasing process which clearly satisfies $\mathfrak{l}_t \rightarrow \infty$ as $t \rightarrow \infty$, since \mathfrak{p} is a positive-recurrent process. The process inverse $\mathfrak{s}_r = \inf\{t \in \mathbb{R}^+ \mid \mathfrak{l}_t \geq r\}$ has independent and stationary increments and is thus an increasing Levy processes. The flats of \mathfrak{l} correspond to excursions from the origin for \mathfrak{p} and jumps for \mathfrak{s} .

We can give a closed expression for the Laplace transform $\mathbb{E}[e^{-\gamma \mathfrak{s}_t}]$. The Laplace transform has the form

$$\mathbb{E}[e^{-\gamma \mathfrak{s}_t}] = e^{-\frac{t}{G_\gamma(0,0)}}. \quad (\text{A.1})$$

where G_γ is the Green function for the Ornstein-Uhlenbeck process. The densities $Q_t : \mathbb{R} \rightarrow \mathbb{R}^+$ for \mathfrak{p}_t satisfy the forward equation

$$\frac{d}{dt} Q_t(p) = \frac{1}{2} Q_t(p) + \frac{1}{2} p \frac{\partial}{\partial p} Q_t(p) + \frac{1}{2} \frac{\partial^2}{\partial^2 p} Q_t(p).$$

When $Q_0(p) = \delta_0(p)$, then $Q_t(p)$ has the explicit form

$$Q_t(p) = \frac{e^{-\frac{p^2}{2\omega_t}}}{(2\pi\omega_t)^{\frac{1}{2}}}, \quad \omega_t = 1 - e^{-\frac{1}{2}t}. \quad (\text{A.2})$$

Notice that there is convergence to a variance-1 Gaussian in the limit that $t \rightarrow \infty$. The form (A.2) allows the Green function value $G_\gamma(0,0)$ to be computed as the following:

$$\begin{aligned} G_\gamma(0,0) &= \int_0^\infty dt e^{-\gamma t} Q_t(0) = (2\pi)^{-\frac{1}{2}} \int_0^\infty dt \frac{e^{-\gamma t}}{(1 - e^{-\frac{1}{2}t})^{\frac{1}{2}}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^1 du u^{2\gamma-1} (1-u)^{-\frac{1}{2}} \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathbf{B}(2\gamma, \frac{1}{2}) = 2^{\frac{1}{2}} \frac{\Gamma(2\gamma)}{\Gamma(2\gamma + \frac{1}{2})}, \end{aligned}$$

where \mathbf{B} and Γ are respectively the β -function and γ -functions, and we have made the substitution $u = e^{-\frac{1}{2}t}$, $-2u^{-1}du = dt$ for the third equality. Plugging our results into (A.1), the moment-generating function of \mathfrak{s}_t is

$$\mathbb{E}[e^{-\gamma \mathfrak{s}_t}] = e^{-t 2^{-\frac{1}{2}} \frac{\Gamma(2\gamma + \frac{1}{2})}{\Gamma(2\gamma)}}.$$

The Levy rate density $R : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ for \mathfrak{s}_t satisfies that

$$\int_0^\infty d\tau (1 - e^{-\gamma\tau}) R(\tau) = 2^{-\frac{1}{2}} \frac{\Gamma(2\gamma + \frac{1}{2})}{\Gamma(2\gamma)}.$$

The rates $R(\tau) = 4^{-1}(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{4}\tau} (1 - e^{-\frac{1}{2}\tau})^{-\frac{3}{2}}$ can be deduced by similar operations as above in reverse order, since

$$\begin{aligned} 2^{-\frac{1}{2}} \frac{\Gamma(2\gamma + \frac{1}{2})}{\Gamma(2\gamma)} &= \frac{2\gamma}{(2\pi)^{\frac{1}{2}}} \mathbf{B}(2\gamma + \frac{1}{2}, \frac{1}{2}) = \frac{\gamma}{(2\pi)^{\frac{1}{2}}} \int_0^\infty d\tau e^{-\gamma\tau} \frac{e^{-\frac{1}{4}\tau}}{(1 - e^{-\frac{1}{2}\tau})^{\frac{1}{2}}} \\ &= \frac{1}{4(2\pi)^{\frac{1}{2}}} \int_0^\infty d\tau (1 - e^{-\gamma\tau}) \frac{e^{-\frac{1}{4}\tau}}{(1 - e^{-\frac{1}{2}\tau})^{\frac{3}{2}}}. \end{aligned}$$

A.2 A diffusion time-changed by \mathbf{l}_t

Now we consider the process $\mathbf{B}_{\mathbf{l}}$ where \mathbf{B} is a Brownian motion with diffusion rate κ which is independent of the process \mathbf{l} discussed in the last section. Although $\mathbf{B}_{\mathbf{l}}$ is non-Markovian, the triple $(\mathbf{B}_{\mathbf{l}}, \tau, \eta)$ is Markovian, where $\tau_t = \mathbf{s}_{\ell_t} - \mathbf{s}_{\ell_{t-}}$ is the total duration of the current excursion (which require some information from the future), and $\eta_t = t - \mathbf{s}_{\ell_{t-}}$ is the amount of time that has passed since the beginning of the excursion.

We can give a closed form for the joint density $\rho_t(x, \tau, \eta)$ for the triple $(\mathbf{B}_{\mathbf{l}_t}, \tau_t, \eta_t)$ assuming that \mathbf{B}_0 has density $\rho(x)$ and $\eta_0 = \tau_0 = 0$. Let $\Psi_r(t)$ be the probability density at the value $t \in \mathbb{R}^+$ for the Levy process \mathbf{s} at time r . The joint density $\rho_t(x, \tau, \eta)$ for the triple $(\mathbf{B}_{\mathbf{l}_t}, \tau_t, \eta_t)$ has the closed form

$$\rho_t(x, \tau, \eta) = \chi(\eta \leq \tau \wedge t) R(\tau) \int_0^\infty dr \Psi_r(t - \eta) (g_r * \rho)(x), \quad g_r(x) = \frac{e^{-\frac{x^2}{2r\kappa}}}{(2\pi r \kappa)^{\frac{1}{2}}},$$

where $R : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the rate function for the Levy process \mathbf{s} . By integrating out the τ, η variables, we attain that the marginal density $\rho_t(x)$ which satisfies the Volterra-type integro-differential equation of the form

$$\rho_t(x) = \rho_0(x) + \frac{\kappa}{2} \int_0^t dr \frac{(2\pi)^{-\frac{1}{2}}}{(1 - e^{-\frac{1}{2}(t-r)})^{\frac{1}{2}}} (\Delta \rho_r)(x), \quad (\text{A.3})$$

where we used that $\Psi_s * \Psi_t = \Psi_{s+t}$ and the explicit computation

$$\int_0^\infty dr \Psi_r(t) = Q_t(0) = \frac{(2\pi)^{-\frac{1}{2}}}{(1 - e^{-\frac{1}{2}t})^{\frac{1}{2}}}.$$

The above is analogous to the master equation for a Brownian motion time-changed by a Mittag-Leffler process. The Mittag-Leffler process $\mathbf{m}^{(\alpha)}$ of index $0 < \alpha < 1$ distributed as the process inverse of the one-sided stable law of index α . The $\alpha = \frac{1}{2}$ case has the same law as the local time of a standard Brownian motion. If \mathbf{B} is a standard Brownian motion, then the densities for $\sqrt{\kappa} \mathbf{B}_{\mathbf{m}_t^{(\alpha)}}$ satisfy the equation

$$\rho_t(x) = \rho_0(x) + \frac{\kappa}{2\Gamma(\alpha)} \int_0^t dr (t - r)^{\alpha-1} (\Delta \rho_r)(x).$$

This is equivalent to the fractional diffusion equation

$$\partial_t^\alpha \rho_t = \kappa \Delta_q \rho_t,$$

where the fractional derivative ∂_t^α acts as $(\partial_t^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t dr (t - r)^{-\alpha} f(r)$. Processes satisfying these equations arise in the theory of continuous time random walks [20, 19] and the limit theory for martingales whose quadratic variations are driven by additive functionals of null-recurrent Markov processes [25, 4, 12]. The process $\mathbf{B}_{\mathbf{m}^{(\alpha)}}$ has the scale invariance in law

$$\mathbf{B}_{\mathbf{m}_t^{(\alpha)}} \stackrel{\mathcal{L}}{=} \epsilon^{-\frac{\alpha}{2}} \mathbf{B}_{\mathbf{m}_{\epsilon t}^{(\alpha)}}.$$

A.3 Long-term behavior

Now we can look into the diffusive behavior for $\mathbf{B}_{\mathbf{l}_t}$ in the limit of large times t . Since the process is already a diffusion, this is just a question of the convergence in probability for the normalized quadratic variation $t^{-1}\mathbf{l}_{st}$ for $s \in \mathbb{R}^+$ as $t \rightarrow \infty$. However, we actually have a strong limit, since

$$\lim_{t \rightarrow \infty} \frac{\mathbf{l}_{st}}{t} = s \lim_{r \rightarrow \infty} \frac{r}{\mathbf{s}_r} = s \left(\int_0^\infty d\tau \tau R(\tau) \right)^{-1} = s(2\pi)^{-\frac{1}{2}}.$$

The first equality holds since \mathbf{l} and \mathbf{s} are process inverses of one another and tend to infinity almost surely. The second equality is the strong law of large numbers for the Levy process \mathbf{s}_r . The computation for the third equality is based on the representation of the Laplace transform of \mathbf{s}_t from the last section. The above implies the convergence in law

$$t^{-\frac{1}{2}}\mathbf{B}_{\mathbf{l}_{st}} \xrightarrow{\mathcal{L}} (2\pi)^{-\frac{1}{2}}\mathbf{B}'_s,$$

where \mathbf{B}' is a copy of standard Brownian motion.

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